Chapter Four

Solving Systems of Non-linear Equations

In this chapter, we consider the problem of finding roots of simultaneous nonlinear equations. For simplicity we shall consider only the case of two equations in two unknowns.

$$\left.\begin{aligned} f(x,\ y) &= 0 \\ g(x,\ y) &= 0 \end{aligned}\right\} \ldots\ldots\ldots(1)$$

We shall study three numerical methods to solve this type of systems:

(i) Fixed-point iteration (for systems):

As a first step in applying fixed-point iteration, we rewrite these equations in the equivalent form

$$\left.\begin{aligned} x &= F(x,\ y) \\ y &= G(x,\ y) \end{aligned}\right\} \ldots\ldots\ldots(2)$$

So that any solution of (2) is a solution of (1), and conversely. Let $\{\lambda,\mu\}$ be a solution of (1), i.e. $\left.\begin{aligned} f(\lambda,\mu) &= 0 \\ g(\lambda,\mu) &= 0 \end{aligned}\right\}$. Let $\{x_o,y_o\}$ be an approximation to $\{\lambda,\mu\}$. Generate successive approximations from the recursion

$$\left.\begin{aligned} x_{i+1} &= F(x_i, y_i) \\ y_{i+1} &= G(x_i, y_i) \end{aligned}\right\},\ i=0,1,2,\ldots \qquad \ldots\ldots\ldots\ldots(3)$$

Stop iteration if $|x_{i+1}-x_i|<\varepsilon$ and $|y_{i+1}-y_i|<\varepsilon$ for any i.

It is shown in analysis this iteration will converge under the following (but not necessary) conditions:

a. F and G and their partial derivatives are continuous in a neighborhood $\Re$ of the roots $\{\lambda, \mu\}$, where $\Re$ consists of all $\{x, y\}$ with $|x-\lambda| \le\varepsilon$, $|y-\mu| \le\varepsilon$, for some positive ε.
b. The following inequalities are satisfied $\left.\begin{aligned} |F_x| + |G_x| &\le K \\ |F_y| + |G_y| &\le K \end{aligned}\right\}$ for all points $\{x,y\}$ in $\Re$ and some $K<1$.
c. The initial approximations $\{x_o, y_o\}$ is chosen in $\Re$.

Note:

When this iteration does converge it converges linearly.

To proof (b):

$$\begin{matrix} x_1 = F(x_0, y_0) \\ y_1 = G(x_0, y_0) \end{matrix} \text{ and } \begin{matrix} \lambda = F(\lambda, \mu) \\ \lambda = G(\lambda, \mu) \end{matrix}.$$

$$x_1 - \lambda = F(x_0, y_0) - F(\lambda,\mu)$$

$$y_1 - \mu = G(x_0, y_0) - G(\lambda,\mu)$$

By suing Taylor series expansion about λ and μ we get

$$F(x_0, y_0) = F(\lambda,\mu) + (x_0 - \lambda)(F_x)_{(x_0,y_0)} + (y_0 - \mu)(F_y)_{(x_0,y_0)} + \cdots$$

$$G(x_0, y_0) = G(\lambda,\mu) + (x_0 - \lambda)(G_x)_{(x_0,y_0)} + (y_0 - \mu)(G_y)_{(x_0,y_0)} + \cdots$$

Let $K = \max\left\{ \left|F_x\right|_{(x_0,y_0)} + \left|G_x\right|_{(x_0,y_0)}, \left|F_y\right|_{(x_0,y_0)} + \left|G_y\right|_{(x_0,y_0)} \right\}$

$$\therefore \quad |x_1 - \lambda| + |y_1 - \mu| \le |x_0 - \lambda| \left\{ \left|F_x\right|_{(x_0,y_0)} + \left|G_x\right|_{(x_0,y_0)} \right\} + |y_0 - \mu| \left\{ \left|F_y\right|_{(x_0,y_0)} + \left|G_y\right|_{(x_0,y_0)} \right\}$$

$$\le K \left\{ |x_0 - \lambda| + |y_0 - \mu| \right\}$$

Similarly,

$$|x_2 - \lambda| + |y_2 - \mu| \le K\left\{ |x_1 - \lambda| + |y_1 - \mu| \right\} \le K^2 \left\{ |x_0 - \lambda| + |y_0 - \mu| \right\}$$

$$\vdots$$

$$|x_n - \lambda| + |y_n - \mu| \le K^n \left\{ |x_0 - \lambda| + |y_0 - \mu| \right\}$$

The value of $|x_0 - \lambda|$ and $|y_0 - \mu|$ must approach to zero and this happen, when K<1 $\{K^n \to 0$ an $n \to \infty\}$. Hence the condition K<1 is sufficient to force the convergence.

Example 1:

Solve the following system:

$0.1x^2+0.1y^2-x+0.8=0$

$0.1x+0.1xy^2-y+0.8=0$ where $x_0=y_0=\frac{1}{5}$.

The exact solution of this system is x=y=1.

Solution:

Rewrite above system as follows:

$x = 0.1x^2+0.1y^2+0.8=F(x,y)$

$y=0.1x+0.1xy^2+0.8=G(x,y)$

We see that $\begin{matrix} |F_x|_{(0.5,\ 0.5)} + |G_x|_{(0.5,\ 0.5)} = 0.144 < 1 \\ |F_y|_{(0.5,\ 0.5)} + |G_y|_{(0.5,\ 0.5)} = 0.48 < 1 \end{matrix}$, hence the iteration converge. Applying iteration form (3) we obtain the successive approximations in this table

i	x_i	y_i
0	0.5	0.5
1	0.85	0.8625
2	0.94664	0.94823
3	0.97953	0.97978
4	0.99182	0.99196

(ii) Newton-Raphson method:

To adapt Newton-Raphson method to simultaneous equations, we proceed as follows: Let $\{x_o,y_o\}$ be an approximation to the solution $\{\lambda,\mu\}$ of (1). Assuming that f and g are sufficiently differentiable, expand f(x,y), g(x,y) about $\{x_o,y_o\}$ using Taylor series for functions of two variables:

$$f(x,y)=f(x_o,y_o)+f_x(x_o,y_o)(x-x_o)+f_y(x_o,y_o)(y-y_o)+\dots$$

$$g(x,y)=g(x_o,y_o)+g_x(x_o,y_o)(x-x_o)+g_y(x_o,y_o)(y-y_o)+\dots$$

Assuming that $\{x_o,y_o\}$ sufficiently closed to $\{\lambda,\mu\}$ so that higher order terms can be neglected, we equate the expansion through linear terms to zero. This gives us the system

$$\left.\begin{matrix} f_x(x-x_o)+f_y(y-y_o) \approx -f \\ g_x(x-x_o)+g_y(y-y_o) \approx -g \end{matrix}\right\} \dots\dots\dots\dots\dots(*)$$

where it is understood that all functions and derivatives in (*) are to be evaluated at $\{x_o,y_o\}$. We might then expect that the solution $\{x_1,y_1\}$of (*) will be closer to the solution $\{\lambda,\mu\}$ than $\{x_o,y_o\}$. The solution of (*) by Cramer's rule yields

$$|x_1 - x_o| = \frac{\begin{vmatrix} -f & f_y \\ -g & g_y \end{vmatrix}}{\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}} = \left[\frac{-fg_y + df_y}{J(f,\ g)}\right]_{\{x_0,y_0\}} \quad ; \quad |y_1 - y_o| = \frac{\begin{vmatrix} f_x & -f \\ g_x & -g \end{vmatrix}}{\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}} = \left[\frac{-gf_x + fg_x}{J(f,\ g)}\right]_{\{x_0,y_0\}}$$

Provided that $J(f,g)=f_xg_y-g_xf_y\neq0$ at $\{x_o,y_o\}$. The function J(f,g) is called the Jacobian of the functions f and g. The solution $\{x_1,y_1\}$of this system now provides a new approximation to $\{\lambda,\mu\}$. Repetition of this process leads to Newton-Raphson method for systems

$$x_{i+1} = x_i - \left[\frac{fg_y - gf_y}{J(f,\ g)}\right]_{\{x_i, y_i\}} ; \qquad y_{i+1} = y_i - \left[\frac{gf_x - fg_x}{J(f,\ g)}\right]_{\{x_i, y_i\}} ;\ i=0,\ 1,\ \ldots$$

Where $J(f,g)=f_x g_y - g_x f_y$ and where all functions involved are to be evaluated at $\{x_i,y_i\}$. Stop iteration if $|x_{i+1}-x_i|<\varepsilon$ and $|y_{i+1}-y_i|<\varepsilon$ for any i.

When this iteration converges, it converges quadratically.

A set of conditions sufficient to ensure convergence is the following:

1. f, g and all their derivatives through second order are continuous and bounded in a region $\Re$ containing $\{\lambda,\mu\}$.
2. The Jacobian J(f,g) dose not vanish in $\Re$.
3. The initial approximation $\{x_o,y_o\}$ is chosen sufficiently close to the root $\{\lambda,\mu\}$.

Example 2:

Solve the system

$x^2+y^2=1$

$x^2-y^2=-0.5$ at $\{x_o,y_o\}=\{0.1,\ 0.3\}$ also at $\{x_o,y_o\}=\{0.5,\ 0.5\}$.

Solution:

Let $f(x,y)=x^2+y^2-1=0$

$g(x,y)=x^2-y^2+0.5=0$

$\therefore$ $f_x=2x,\ f_y=2y,\ g_x=2x,\ g_y=-2y$

at $\{x_o,y_o\}=\{0.1,\ 0.3\} \Rightarrow$ $f(0.1,0.3)=-0.9$, $f_x=0.2$, $f_y=0.6$, $g(0.1,0.3)=0.42$, $g_x=0.2$, $g_y=-0.6$

$$\therefore \quad x_1 = x_0 - \left[\frac{fg_y - gf_y}{J(f,\ g)}\right]_{\{x_0, y_0\}} = 0.1 - \left[\frac{(-0.9\times -0.6)-(0.42\times 0.6)}{(0.2\times -0.6)-(0.6\times 0.2)}\right] = 0.1 - \frac{0.288}{-0.24} = 1.3$$

$$y_1 = y_0 - \left[\frac{gf_x - fg_x}{J(f,\ g)}\right]_{\{x_0, y_0\}} = 0.3 - \left[\frac{(0.42\times 0.2)-(-0.9\times 0.2)}{(0.2\times -0.6)-(0.6\times 0.2)}\right] = 0.3 - \frac{0.264}{-0.24} = 1.4$$

Also

$$x_2 = x_1 - \left[\frac{fg_y - gf_y}{J(f,\ g)}\right]_{\{x_1, y_1\}} = 1.3 - \left[\frac{fg_y - gf_y}{J(f,\ g)}\right]_{\{1.3,\ 1.4\}} = ?$$

$$y_2 = y_1 - \left[\frac{gf_x - fg_x}{J(f,\ g)}\right]_{\{x_1, y_1\}} = 1.4 - \left[\frac{gf_x - fg_x}{J(f,\ g)}\right]_{\{1.3,\ 1.4\}} = ?$$

$\vdots$

(iii) Modified Newton-Raphson method:

Newton-Raphson method is not very easy in general for n simultaneous equations in n unknowns. But in Modified Newton-Raphson method we sue the idea of Newton-Raphson method for single variable as follows:

For nonlinear system $\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases}$

$$x_{i+1} = x_i - \left[\frac{f}{f_x}\right]_{\{x_i, y_i\}} ; \qquad y_{i+1} = y_i - \left[\frac{g}{g_y}\right]_{\{x_{i+1}, y_i\}} \text{; i=0, 1, ...}$$

Stop iteration if $|x_{i+1}-x_i|<\varepsilon$ and $|y_{i+1}-y_i|<\varepsilon$ for any i.

Also for nonlinear system $\begin{cases} f(x, y, z) = 0 \\ g(x, y, z) = 0 \\ h(x, y, z) = 0 \end{cases}$

$$x_{i+1} = x_i - \left[\frac{f}{f_x}\right]_{\{x_i, y_i, z_i\}} ; \quad y_{i+1} = y_i - \left[\frac{g}{g_y}\right]_{\{x_{i+1}, y_i, z_i\}} ; \quad z_{i+1} = z_i - \left[\frac{h}{h_z}\right]_{\{x_{i+1}, y_{i+1}, z_i\}} ;$$

For i=0, 1, …

Stop iteration if $|x_{i+1}-x_i|<\varepsilon$, $|y_{i+1}-y_i|<\varepsilon$ and $|z_{i+1}-z_i|<\varepsilon$ for any i.