Chapter five

1.1 The finite difference calculus:

Given a discrete function $f(x_k)=y_k$ that is each arguments x_k has a mate y_k and suppose that the arguments are equally spaced so that $x_{k+1}-x_k=h$. Then we define the following difference operators of the y_k.

1.1.1) Shifting operator (E):

This operator is defined as $Ef(x)=f(x+h)$ i.e. $Ey_0=y_1$

$E^2f(x)=f(x+2h)$ i.e. $E^2y_0=y_2$

$\vdots$

$E^kf(x)=f(x+kh)$ i.e. $E^ky_0=y_k$. In general $E^ky_i=y_{i+k}$ for i=0, 1, …; k=1, 2, …

1.1.2) Forward difference operator (Δ):

This operator is defined as $\Delta f(x_0)=f(x_0+h)-f(x_0)$ or $\Delta y_k=y_{k+1}-y_k$ where k=0, 1, 2, ... i.e. $\Delta y_0=y_1-y_0$ or $\Delta y_0=Ey_0-y_0=(E-1)\,y_0 \Rightarrow \Delta=E-1$.

The difference $\Delta y_k=y_{k+1}-y_k$ is called first difference and the difference of the second difference is denoted by $\Delta^2 y_k=\Delta\,(\Delta y_k)=\Delta(y_{k+1}-y_k)=\Delta\, y_{k+1}-\Delta\, y_k=y_{k+2}-2y_{k+1}+y_k$.

In general $\Delta^n y_i=\sum_{j=0}^{n}(-1)^j\binom{n}{j}y_{i+n-j}$ or $\Delta^n f(x)=\sum_{j=0}^{n}(-1)^j\binom{n}{j}f(x+jh)$

where $\binom{n}{j}=\dfrac{n!}{j!(n-j)!}$

Note:

i) If $f(x)=c$ then $\Delta f(x)=0$. Because $\Delta f(x)=f(x+h)-f(x)=c-c=0$.

ii) If $f(x)=ax^2+bx+c$ then $\Delta^2 f(x)=2ah^2$ and $\Delta^3 f(x)=0$.

Because: $\Delta f(x)=f(x+h)-f(x)=a(x+h)^2+b(x+h)+c-ax^2-bx-c=2axh+ah^2+bh$

$\Delta^2 f(x)=\Delta f(x+h)-\Delta f(x)=2a(x+h)h+ah^2+bh-2axh-ah^2-bh=2ah^2$

$\Delta^3 f(x)=0$ by (i).

Exs: Show that if $p(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ then $\Delta^n p(x)=n!a_nh^n$ and $\Delta^{n+1}p(x)=0$.

Forward difference table

From the above, we can form the following forward difference table

X	f(x)	Δ	Δ^2	Δ^3	Δ^4	…..
$\vdots$	$\vdots$					
x_{-2}	y_{-2}					
		$\Delta y_{-2}=y_{-1}-y_{-2}$				
x_{-1}	y_{-1}		$\Delta^2 y_{-2}=\Delta y_{-1}-\Delta y_{-2}$			
		$\Delta y_{-1}=y_0-y_{-1}$		$\Delta^3 y_{-2}=\Delta^2 y_{-1}-\Delta^2 y_{-2}$		
x_0	y_0		$\Delta^2 y_{-1}=\Delta y_0-\Delta y_{-1}$		$\Delta^4 y_{-2}=\Delta^3 y_{-1}-\Delta^3 y_{-2}$	
		$\Delta y_0=y_1-y_0$		$\Delta^3 y_{-1}=\Delta^2 y_0-\Delta^2 y_{-1}$		
x_1	y_1		$\Delta^2 y_0=\Delta y_1-\Delta y_0$		$\Delta^4 y_{-1}=\Delta^3 y_0-\Delta^3 y_{-1}$	
		$\Delta y_1=y_2-y_1$		$\Delta^3 y_0=\Delta^2 y_1-\Delta^2 y_0$		
x_2	y_2		$\Delta^2 y_1=\Delta y_2-\Delta y_1$			
		$\Delta y_2=y_3-y_2$				
x_3	y_3					
$\vdots$	$\vdots$					

Example: construct the forward difference table for the following values:

(i) (0,1), (1,5), (2,31), (3,121) and (4,341)

(ii) (1,0), (2,5), (3,22), (4,57), (5,116), (6,205).

Solution:

(i):

x	f(x)	Δ	Δ^2	Δ^3	Δ^4
0	1				
		4			
1	5		22		
		26		42	
2	31		64		24
		90		66	
3	121		130		
		220			
4	341				

(ii)

x	f(x)	Δ	Δ^2	Δ^3	Δ^4
1	0				
		5			
2	5		12		
		17		6	
3	22		18		0
		35		6	
4	57		24		0
		59		6	
5	116		30		
		89			
6	205				

1.1.3) Backward difference operator (∇):

This operator is defined as $\nabla y_i=y_i-y_{i-1}$; i=1, 2, …

$\nabla^2 y_i=\nabla(\nabla y_i)= \nabla(y_i-y_{i-1})= \nabla y_i-\nabla y_{i-1}=(y_i-y_{i-1})-(y_{i-1}-y_{i-2})= y_i-2y_{i-1}+y_{i-2}$

In general $\nabla^n y_i=\sum_{j=0}^{n}(-1)^j\binom{n}{j}y_{i-j}$.

Exercise: Show that $\nabla^i y_0=\Delta^i y_{-1}$.

Backward difference table:

x	f(x)	∇	∇^2	∇^3	∇^4	……….
⋮	⋮					
x_{-2}	y_{-2}					
		$\nabla y_{-1}=y_{-1}-y_{-2}$				
x_{-1}	y_{-1}		$\nabla^2 y_0=\nabla y_0-\nabla y_{-1}$			
		$\nabla y_0=y_0-y_{-1}$		$\nabla^3 y_1=\nabla^2 y_1-\nabla^2 y_0$		
x_0	y_0		$\nabla^2 y_1=\nabla y_1-\nabla y_0$		$\nabla^4 y_2=\nabla^3 y_2-\Delta^3 y_1$	
		$\nabla y_1=y_1-y_0$		$\nabla^3 y_2=\Delta^2 y_2-\nabla^2 y_1$		
x_1	y_1		$\nabla^2 y_2=\Delta y_2-\Delta y_1$		$\nabla^4 y_3=\nabla^3 y_3-\nabla^3 y_2$	
		$\nabla y_2=y_2-y_1$		$\nabla^3 y_3=\nabla^2 y_3-\nabla^2 y_2$		
x_2	y_2		$\nabla^2 y_3=\nabla y_3-\nabla y_1$			
		$\nabla y_3=y_3-y_2$				
x_3	y_3					
⋮	⋮					

Exercises:

(1): show that $\Delta(u(x)v(x))=u(x)\,\Delta v(x)+v(x+h)\,\Delta u(x)$ Or $\Delta u_i v_i=u_i\Delta v_i+v_{i+1}\Delta u_i$.

Solution:

$$\Delta(u(x)v(x))=u(x+h)v(x+h)-u(x)v(x)= u(x+h)v(x+h)-u(x)v(x)-(x+h)u(x)+v(x+h)u(x)$$

$$=v(x+h)[u(x+h)-u(x)]+u(x)[v(x+h)-v(x)]= u(x)\,\Delta v(x)+v(x+h)\,\Delta u(x).$$

(2): Show that:

(i) $\Delta\left(\frac{u(x)}{v(x)}\right)=\frac{v(x)\Delta u(x)-u(x)\Delta v(x)}{v(x+h)v(x)}$ (ii) $\sum_{i=0}^{n-1}\Delta y_i=y_n-y_0$

(iii) $\sum_{i=0}^{n-1}u_i\Delta v_i=u_n v_n-u_0 v_0-\sum_{i=0}^{n-1}v_{i+1}\Delta u_i$ (iv) $y_k=\sum_{i=0}^{k}\binom{k}{i}\Delta^i y_0$ (using mathematical induction)

(vii) $\Delta^n f(x)=\frac{\Delta^n f(x)}{n!h^n}$ (using mathematical induction) (viii) $\Delta(u_i+v_i)=\Delta u_i+\Delta v_i$

(ix) $\Delta(c_1u_i+c_2v_i)=c_1\Delta u_i+c_2\Delta v_i$ (x) $\Delta\sin(k)=2\cos(k+\frac{1}{2})\sin(\frac{1}{2})$.

Solution:

$$\Delta\sin(k)=\sin(k+1)-\sin(k)=\sin(k+\frac{1}{2}+\frac{1}{2})-\sin(k)=\sin(k+\frac{1}{2})\cos(\frac{1}{2})+\cos(k+\frac{1}{2})\sin(\frac{1}{2})-$$

$$\sin(k)=\sin(k+\frac{1}{2})\cos(\frac{1}{2})+\cos(k+\frac{1}{2})\sin(\frac{1}{2})-\sin(k+\frac{1}{2}-\frac{1}{2})=\sin(k+\frac{1}{2})\cos(\frac{1}{2})+\cos(k+\frac{1}{2})\sin(\frac{1}{2})-(\sin(k+\frac{1}{2})\cos(\frac{1}{2})-\cos(k+\frac{1}{2})\sin(\frac{1}{2}))=2\cos(k+\frac{1}{2})\sin(\frac{1}{2}).$$

(xi) $\Delta\cos(k)=-2\sin(k+\frac{1}{2})\sin(\frac{1}{2})$ (xii) $\delta=E^{1/2}-E^{-1/2}$ (xiii) $E=1+\Delta$

(xiv) $E\Delta=\Delta E$ (xv) $E\nabla=\nabla E=\Delta$. (xvi) $\nabla=1-E^{-1}$ (xvii) $EE^{-1}=1$.

1.1.6) Divided difference operator (Δ):

Given a discrete function $f(x_k)=y_k$ that is each arguments x_k has a mate y_k and suppose that the arguments x_k, k=0, 1, … . Then we define Δ as follows:

$$\Delta y_i=\frac{y_{i+1}-y_i}{x_{i+1}-x_i};\ i=0, 1, \ldots$$

$$\Delta^2 y_i=\frac{\Delta y_{i+1}-\Delta y_i}{x_{i+2}-x_i}=\frac{\frac{y_{i+2}-y_{i+1}}{x_{i+2}-x_{i+1}}-\frac{y_{i+1}-y_i}{x_{i+1}-x_i}}{x_{i+2}-x_i}$$

In general $\Delta^k y_i=\dfrac{\Delta^{k-1}y_{i+1}-\Delta^{k-1}y_i}{x_{i+k}-x_i}$; i=0, 1, …, n-k

Divided deference table:

x	y	Δ	Δ^2	Δ^3	…
x_0	y_0				
		$\Delta y_0=\frac{y_1-y_0}{x_1-x_0}$			
x_1	y_1		$\Delta^2 y_0=\frac{\Delta y_1-\Delta y_0}{x_2-x_0}$		
		$\Delta y_1=\frac{y_2-y_1}{x_2-x_1}$		$\Delta^3 y_0=\frac{\Delta^2 y_1-\Delta^2 y_0}{x_3-x_0}$	
x_2	y_2		$\Delta^2 y_1=\frac{\Delta y_2-\Delta y_1}{x_3-x_1}$		
		$\Delta y_2=\frac{y_3-y_2}{x_3-x_2}$		$\Delta^3 y_1=\frac{\Delta^2 y_2-\Delta^2 y_1}{x_4-x_1}$	
x_3	y_3		$\Delta^2 y_2=\frac{\Delta y_3-\Delta y_2}{x_4-x_2}$		

$$\Delta y_3 = \frac{y_4 - y_3}{x_4 - x_3}$$

x_4 y_4

⋮ ⋮

We also define divided difference as follows:

$$f[x_i] = f(x_i)$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} = \Delta y_i$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i} = \frac{\frac{f[x_{i+2}] - f[x_{i+1}]}{x_{i+2} - x_{i+1}} - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}}{x_{i+2} - x_i}$$

$$= \frac{\frac{y_{i+2} - y_{i+1}}{x_{i+2} - x_{i+1}} - \frac{y_{i+1} - y_i}{x_{i+1} - x_i}}{x_{i+2} - x_i} = \Delta^2 y_i$$

In general $f[x_i, x_{i+1}, ..., x_{i+n}] = \dfrac{f[x_{i+1}, ..., x_{i+n}] - f[x_i, x_{i+1}, ..., x_{i+n-1}]}{x_{i+n} - x_i} = \Delta^n y_i$

Note:

$f[x_i, x_{i+1}] = f[x_{i+1}, x_i]$, $f[x_i, x_j, x_k] = f[x_j, x_i, x_k] = f[x_j, x_k, x_i] = ...$

1.2 Interpolation:

Interpolation is a method used in numerical analysis to approximate functions or to estimate the value of a function f(x) for arguments between x_0, x_1, …,x_n at which the values y_0, y_1,…,y_n are known. The goal of this method is to replace a given function (whose values are known at determined points) by another one which is simpler. Interpolation has many applications: We know its values at specific points, approximating the integral and derivatives of function, and numerical solutions of integral and differential equation, the most used functions in interpolation, are polynomials, trigonometric, exponentials, and rationals. We will only consider here interpolation by polynomials.

1.2.1The interpolation Problem:

Let x_0, x_1, …,x_n be (n+1) distinct points on the x-axis, and f(x) be a real-valued function defined on [a,b] such that

$a \le x_0 < x_1 < \dots < x_n \le b$ (1.1)

we suppose known the values of f at these points. Let

$y_i = f(x_i)$, i=0, 1, …, n (1.2)

We want to prove the existence and uniqueness of a polynomial $p_n(x)$ of degree≤n which interpolates (takes the same values as) f(x) at the given (n+1) distinct points. That is it satisfies

$p_n(x_i) = y_i = f(x_i)$, i=0, 1, …, n (1.3)

This polynomial will be constructed and called the interpolation polynomial.

1.2.1. 1 Lagrange Interpolation polynomial:

Suppose that a polynomial

$p_n(x)=a_nx^n+a_{n-1}x^{n-1}+\dots+a_1x+a_0$ (1.4)

of degree n satisfies (1.3). Then, the condition that this polynomial must pass through this (n+1) points leads to (n+1) equations for the (n+1) unknown's a_i as follows:

$$f(x_0) = a_n x_0^n + a_{n-1}x_0^{n-1} + \dots + a_1 x_0 + a_0$$
$$f(x_1) = a_n x_1^n + a_{n-1}x_1^{n-1} + \dots + a_1 x_1 + a_0$$
$$\vdots$$
$$f(x_n) = a_n x_n^n + a_{n-1}x_n^{n-1} + \dots + a_1 x_n + a_0$$

We find the values of a_i's by solving above linear system of equations. Then, put the value of a_i's in (1.4) and add the coefficients, we get

$$p_n(x) = \sum_{k=0}^{n} L_n^k(x) f(x_k) \quad (1.5)$$

Where

$$L_n^k(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{k-1})(x-x_{k+1})\dots(x-x_n)}{(x_k-x_0)(x_k-x_1)\dots(x_k-x_{k-1})(x_k-x_{k+1})\dots(x_k-x_n)} = \prod_{\substack{i=0\\ i\ne k}}^{n}\left(\frac{x-x_i}{x_k-x_i}\right) \quad (1.6)$$

Such that

$$L_n^k(x_i) = \delta_{ki} = \begin{cases} 0 & \text{if } k \ne i \\ 1 & \text{if } k = i \end{cases} \quad (1.7)$$

The polynomials in (1.6) are called ***Lagrange polynomial*** and (1.5) is of degree≤n and is called ***Lagrange interpolation polynomial***.

To compute $L_n^k(x)$, k=0, 1, 2, …, n by another way. $L_n^k(x)$ is a polynomial of degree n and by (1.7) vanishes at the n distinct points

$x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots x_n$

So it is of the form

$$L_n^k(x) = \alpha(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n).$$

To determine the constant α, we use the condition $L_n^k(x_k)=1$ given by (1.7) to obtain for k=0, 1, 2, …, n the Lagrange polynomial (1.6) which together with (1.5) defines a polynomial of degree ≤n interpolating f at the (n+1) distinct points.

If now $p_n(x)$ and $q_n(x)$ are two polynomials of degree ≤n, interpolating f at the (n+1) distinct points given by (1.1) then $p_n(x_i) = y_i = q_n(x_i)$, i=0, 1, 2, …, n.

It follows then that the polynomial $d_n(x) = p_n(x) - q_n(x)$ which is of degree≤n has (n+1) distinct roots (because $d_n(x_i) = p_n(x_i) - q_n(x_i) = y_i - y_i = 0$, i=0, 1, …, n). Thos is impossible unless $d_n(x)$ vanishes identically, but if $d_n(x)$ vanishes identically, then $p_n(x) = q_n(x)$.

We have proved the existence and uniqueness of a polynomial $p_n(x)$, given by (1.5) and (1.6) of degree ≤n which interpolates f(x) at (n+1) distinct points [i.e. it satisfies (1.3)].

Example: To calculate the interpolating polynomial $p_2(x)$ of the function f such that

x	-1	0	2
y=f(x)	2	-1	5

From (1.5), and (1.6)

$$p_2(x) = \sum_{k=0}^{2} L_2^k(x) f(x_k) = L_2^0(x) f(x_0) + L_2^1(x_1) f(x_1) + L_2^2(x) f(x_2)$$

$$= 2L_2^0(x) - L_2^1(x_1) + 5L_2^2(x)$$

Where

$$L_2^0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0)(x-2)}{(-1-0)(-1-2)} = \frac{1}{3}(x^2-2x)$$

$$L_2^1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x+1)(x-2)}{(0+1)(0-2)} = -\frac{1}{2}(x^2-x-2)$$

$$L_2^2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x+1)(x-0)}{(2+1)(2-0)} = \frac{1}{6}(x^2+x)$$

Hence $p_2(x) = 2x^2 - x - 1$.

To estimate the value of f(x) at 0.5 i.e. f(0.5) from above table, we put this value in p_2(x) we get $f(0.5) \approx p_2(x) = 2(0.5)^2 - (0.5) - 1 = -0.5$

Also from above table find f(-0.4) and f(1).

Error of the polynomial interpolation:

Theorem: Let $f \in C^n[a,b]$ such that $f^{(n+1)}$ exists in (a,b). If p_n(x) is the interpolating polynomial (1.5) of f at the (n+1) distinct points $a \le x_0 < x_1 < \ldots < x_n \le b$ then for any x in [a,b], there exists c∈(a,b) with

$$E_n(x) = f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)\, w(x) \qquad (1.8)$$

Where $w(x) = \prod_{i=0}^{n} (x - x_i)$.

Proof: If $x = x_i$, then $f(x_i) = p_n(x_i)$, $w(x_i)=0$ and (1.8) holds. Fix $x \in [a,b]$, $x \ne x_i$ (i=0,1, 2, …,n) and consider the function

$$k(x) = \frac{f(x) - p_n(x)}{w(x)} \qquad (1.9)$$

And the real-values function $g : [a,b] \to \Re$ of the variable *t*

$$g(t) = f(t) - p_n(t) - (t - x_0)(t - x_1)\ldots(t - x_n)k(x)$$

We have $g \in C^n[a,b]$ and $g^{(n+1)}$ exists in (*a*,*b*), and *g* has at least the (*n*+2) distinct roots x_0, x_1, …,x_n, *x*. It follows then by successive applications of Rolle's theorem (If $f \in C[a,b]$ and is differentiable on (a,b) and $f(a) = f(b)$, then there exists at least one $c \in (a,b)$ such that $f'(c) = 0$) on *g* and its derivatives that $g^{(n+1)}$ has at least one root, say $c \in (a,b)$. Therefore $g^{(n+1)}(c) = f^{(n+1)}(c) - (n+1)!\,k(x) = 0$ which, together with (1.9) implies (1.8).

1.2.1.2 Divided deference interpolation formula:

Methods for determining the explicit representation of an interpolating polynomial from tabulated data are known as **divided difference method**. These methods were widely used for computational purposes before digital computing equipment became readily available. However, the methods can be used to derive techniques for approximating the derivatives and integrals of functions, as well as for approximating the solutions to differential equations.

Our treatment of divided difference methods will be brief since the results in this section will not be used extensively in subsequent material.

Suppose that $p_n(x)$ is Lagrange interpolation polynomial of degree ay most n that agree with the function at the distinct numbers $x_0, x_1, \ldots, x_n$. The divided differences of f with respect to $x_0, x_1, \ldots, x_n$ can be derived by showing that p_n has the representation

$$p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \quad (1.13)$$

For appropriate constants $a_0,\ a_1, \cdots, a_n$.

To determine the first of these constants, a_0, note that if $p_n(x)$ can be written in the form of equation (1.13), then evaluating p_n at x_0 leaves only the constant term a_0; that is, $a_0 = p_n(x_0) = f(x_0)$.

Similarly, when p_n is evaluated at x_1, the only nonzero terms in the evaluation of $p_n(x_1)$ are the constant and linear terms

$$f(x_0) + a_1(x - x_0) = p_n(x_1) = f(x_1)$$

So

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}. \quad (1.14)$$

At this stage we introduce what is known as the divided difference notation. The zeroth divided difference of the function f, with respect to x_1, is denoted by $f[x_1]$ and is simply the evaluation of f an x_1,

$$f[x_1] = f(x_1).$$

The remaining divided difference are defined inductively, the first divided difference of f with respect to x_i and x_{i+1} is denoted by $f[x_i, x_{i+1}]$ and defined as

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}. \quad (1.15)$$

When the (k-1)st divided difference

$f[x_i, x_{i+1}, \ldots, x_{i+k-1}]$ and $f[x_{i+1}, x_{i+2}, \ldots, x_{i+k-1}, x_{i+k}]$

Have both been determined, the kth divided difference relative to $x_i,\ x_{i+1},\ \ldots, x_{i+k}$ is given by

$$f[x_i, x_{i+1},..., x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1},..., x_{i+k}] - f[x_i, x_{i+1},..., x_{i+k-1}]}{x_{i+k} - x_i}. \qquad (1.16)$$

With this notation, equation (1.14) can be re-expressed as $a_1 = f[x_0, x_1]$ and the interpolation polynomial in equation (1.13) is:

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + a_2(x - x_0)(x - x_1) + ... + a_n(x - x_0)(x - x_1)...(x - x_{n-1}).$$

The constants $a_2, a_3, \cdots a_n$ in p_n can be consecutively obtained in a manner similar to the evaluation of a_0 and a_1, but the algebraic manipulation becomes tedious. As might be expected from the evaluation of a_0 and a_1, the required constants are

$a_k = f[x_0, x_1,..., x_{k-1}, x_k]$ for each k=0, 1, …,n; so p_n can be written as

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + ... +$$

$$f[x_0, x_1,..., x_n](x - x_0)(x - x_1)...(x - x_{n-1}) \qquad (1.17)$$

Equation (1.17) is known as ***divided difference interpolation formula***.

Theorem: Suppose that $f \in C^n[a,b]$ and x_0, x_1, …,x_n are distinct numbers in [*a*,*b*]. Then a number ζ in (*a*,*b*) exists with

$$f[x_0, x_1,..., x_n] = \frac{f^{(n)}(\zeta)}{n!}.$$

Proof: Let

$$g(x) = f(x) - p_n(x).$$

Since $f(x_i) - p_n(x_i) = 0$ for each *i*=0, 1, …, *n*, *g* has *n*+1 distinct zeros in [*a*,*b*]. The generalized Rolle's theorem implies that a number ζ in (*a*,*b*) exists with $g^{(n)}(\zeta) = 0$, so

$$0 = f^{(n)}(\zeta) - p_n^{(n)}(\zeta)$$

Since $p_n(x)$ is a polynomial of degree n whose leading coefficient is $f[x_0, x_1,..., x_n]$,

$$p_n^{(n)}(x) = f[x_0, x_1,..., x_n].n!$$

As a consequence, $f[x_0, x_1,..., x_n] = \dfrac{f^{(n)}(\zeta)}{n!}$.

When x_0, x_1, …,x_n are arranged consequently with equal spacing, equation (1.17) can be expressed in a simplified form. Introducing the notation $h=x_{i+1}-x_i$ for each *i*=0, 1, …, *n*-1 and $x=x_0+sh$, the difference $x-x_i$ can be written as $x-x_i=(s-i)h$; so equation (1.17) becomes

$$p_n(x) = p_n(x_0 + sh) = f[x_0] + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2] + \dots +$$

$$s(s-1)\dots(s-n+1)h^n f[x_0, x_1, \dots, x_n]$$

$$= f(x_0) + \sum_{k=1}^{n} s(s-1)\dots(s-k+1)h^k f[x_0, x_1, \dots, x_k].$$

Using binomial-coefficient notation

$$\binom{s}{x} = \frac{s(s-1)\dots(s-k+1)}{k!}$$

We can express $p_n(x)$ compactly as

$$p_n(x) = p_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!h^k f[x_0, x_1, \dots, x_k]. \qquad (1.18)$$

Example: Approximate $f(1.1)$ using the following data and the divided difference interpolation formula:

x	1	1.3	1.6	1.9	2.2
$f(x)$	0.751977	0.200860	0.4554022	0.2818186	0.1103623

Solution: The divided difference table corresponding to this data is given below:

x	$f(x)$	First divided difference	Second divided difference	Third divided difference	Fourth divided difference
1	0.751977				
		-0.4837057			
1.3	0.200860		-0.1087339		
		-0.5489460		0.0658784	
1.6	0.4554022		-0.0494433		0.0018251
		-0.5786120		0.068085	
1.9	0.2818186		-0.0118183		
		-0.5715210			
2.2	0.1103623				

From (1.18) for n=4, we obtain

$$p_4(x) = f(x_0) + \sum_{k=1}^{4} \binom{s}{k} k!h^k f[x_0, x_1, \dots, x_k] = f(x_0) + shf[x_0, x_1] + s(s-1)h^2 f[x_0, x_1, x_2]$$

$$+ s(s-1)(s-2)h^3 f[x_0, x_1, x_2, x_3] + + s(s-1)(s-2)(s-3)h^4 f[x_0, x_1, x_2, x_3, x_4]$$

If x=1.1, this implies that h=0.3 and $s=\frac{1}{3}$. Hence

$$p_4(x) = 0.7651997+\frac{1}{3}(0.3)(-0.4837057)+ \frac{1}{3}(-\frac{2}{3})(0.3)^2(-0.1087339)+$$

$$\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(0.3)^3(0.0658784)+\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})(0.3)^4(0.0018251)$$

$$=0.7196480$$

1.3 Interpolation at equally spaced points (nodes)

1.3.1 Newton formulas for the interpolating polynomial (Newton forward difference interpolation formula (NFDIF)):

We suppose the (n+1) points $x_0, x_1, \ldots, x_n$ to be equally spaced points, with

$$x_{i+1} - x_i = h \;;\; i=0, 1, \ldots, n-1 \tag{1.19}$$

That is

$$x_i = x_0 + ih \;;\; i=0, 1, \ldots, n \tag{1.20}$$

We wild like to be able deduce $p_k(x)$, the polynomial of degree$\leq k$ which interpolates f at the (k+1) points $x_0, x_1, \ldots, x_k$ from $p_{k-1}(x)$: the one which interpolates f at the k points $x_0, x_1, \ldots, x_{k-1}$. We let

$$p_0(x) = f(x_0)$$

And
$$p_k(x) = p_{k-1}(x) + A_k q_k(x) \tag{1.21}$$

Where A_k is a constant and $q_k(x)$ is a polynomial of degree k such that the coefficients of the term in x^k is one. A_k and q_k are to be determined. Since

$$p_k(x_i) = p_{k-1}(x_i) = f(x_i)\,,\; i=0, 1, \ldots, k-1$$

It follows from (1.21) (if $A_k \neq 0$) that q_k has the k distinct roots $x_0, x_1, \ldots, x_{k-1}$.

Therefore

$$q_k(x) = (x-x_0)(x-x_1)\ldots(x-x_{k-1}) \tag{1.22}$$

To determine A_k, we use the additional condition $p_k(x_k) = f(x_k)$ which is equivalent to

$$\Delta^k p_k(x_0) = \Delta^k f(x_0) \tag{1.23}$$

Since, by section (1.1.2),

$$\Delta^k p_k(x_0) = \sum_{j=0}^{k} c_j p_k(x_0 + jh) = \sum_{j=0}^{k} c_j p_k(x_j) = c_k p_k(x_k) + \sum_{j=0}^{k-1} c_j p_k(x_j)$$

And

$$\Delta^k f(x_0) = \sum_{j=0}^{k} c_j f(x_0 + jh) = \sum_{j=0}^{k} c_j f(x_j) = c_k f(x_k) + \sum_{j=0}^{k-1} c_j f(x_j)$$

But form (1.21)

$$\Delta^k p_k(x_0) = \Delta^k p_{k-1}(x_0) + A_k \Delta^k q_k(x_0)$$

Which, using the fact ($\Delta^n p(x) = n! a_n h^n$ and $\Delta^{n+1} p(x) = 0$) and (1.23), implies

$$\Delta^k f(x_0) = A_k k! h^k .$$

Hence

$$A_k = \frac{\Delta^k f(x_0)}{k! h^k} \tag{1.24}$$

Therefore (1.21) becomes

$$p_k(x) = p_{k-1}(x) + \frac{\Delta^k f(x_0)}{k! h^k}(x - x_0)(x - x_1)\ldots(x - x_{k-1})$$

Finally we have the ***Newton forward formula*** (Newton forward difference interpolation formula)

$$p_n(x) = f(x_0) + \frac{\Delta f(x_0)}{h}(x - x_0) + \frac{1}{2!}\frac{\Delta^2 f(x_0)}{h^2}(x - x_0)(x - x_1) + \ldots +$$

$$\frac{1}{n!}\frac{\Delta^n f(x_0)}{h^n}(x - x_0)(x - x_1)\ldots(x - x_{n-1}) . \tag{1.25}$$

Form (1.25) let

$$s = \frac{x - x_0}{h} \tag{1.26}$$

Then we have

$$x - x_i = x - x_0 - (x_i - x_0) = sh - ih = (s - i)h \text{ , i=0,1, …, n.} \tag{1.27}$$

With this, the Newton forward formula (1.25) becomes

$$p_n(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \ldots + \frac{s(s-1)(s-2)\ldots(s-n+1)}{n!}\Delta^n f(x_0) \tag{1.28}$$

and the corresponding error (1.8) becomes

$$E_n(x) = f(x) - p_n(x) = \frac{s(s-1)(s-2)\ldots(s-n)}{(n+1)!} h^{n+1} f^{(n+1)}(c) \tag{1.29}$$

Where $a<c<b$.

Example1.31:

From the following table

x	1	2	3	4	5	6
f(x)	0	5	22	57	116	205

Find $f(2.3)$ and $f(3.5)$.

Solution:

x	f(x)	Δ	Δ^2	Δ^3	Δ^4
1	0				
		5			
2	5		12		
		17		6	
3	22		18		0
		35		6	
4	57		24		0
		59		6	
5	116		30		
		89			
6	205				

For $f(2.3)$, $x_0=2$, $h=1 \Rightarrow$ $s = \frac{x - x_0}{h} = \frac{2.3 - 2}{1} = 0.3$, $y_0 = 5$, $\Delta y_0 = 17$, $\Delta^2 y_0 = 18$,

$\Delta^3 y_0 = 6$ and $\Delta^4 y_0 = 0$

Form (1.28) we have

$$p_3(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0)$$

$$\therefore \quad f(2.3) \cong p_3(2.3) = 5 + (0.3)(17) + \frac{(0.3)(0.3-1)}{2!}18 + \frac{(0.3)(0.3-1)(0.3-2)}{3!}6 = 8.567 \; .$$

For f(3.5), $x_0=3$, $h=1 \Rightarrow$ $s = \frac{x - x_0}{h} = \frac{3.5 - 3}{1} = 0.5$, $y_0 = 22$, $\Delta y_0 = 35$, $\Delta^2 y_0 = 24$,

$\Delta^3 y_0 = 6$ and $\Delta^4 y_0 = 0$

Form (1.28) we have

$$p_3(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)}{3!}\Delta^3 f(x_0)$$

$$\therefore\ f(3.5) \cong p_3(3.5) = 22 + (0.5)(35) + \frac{(0.5)(0.5-1)}{2!}24 + \frac{(0.5)(0.5-1)(0.5-2)}{3!}6 = 36.875\ .$$

If s=0.3:

$$\left|E_3(x)\right| = \left|f(x) - p_3(x)\right| = \left|\frac{s(s-1)(s-2)(s-3)}{4!}h^4 f^{(4)}(c)\right| = \frac{\left|s(s-1)(s-2)(s-3)\right|}{4!}h^4\left|f^{(4)}(c)\right| =$$

$$\frac{\left|0.3(0.3-1)(0.3-2)(0.3-3)\right|}{24}M_4 = 0.0402\quad M_4 \text{ where } \max_{1<c<6}\left|f^{(4)}(c)\right|$$

To show that equation (1.28) is valid when s is rational number:

Let f is continuously differentiable function for any order, then

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots = \left\{1 + hd + \frac{h^2D^2}{2!} + \cdots\right\}f(x_0) = e^{hd}f(x_0)\ , (*)$$

Where $D = \dfrac{d}{dx}$.

But $\Delta f(x_0) = f(x_0+h) - f(x_0) \Rightarrow f(x_0+h) = f(x_0) + \Delta f(x_0) = (1+\Delta)f(x_0)$ (**)

From (*) and (**) we get $1+\Delta = e^{hd}$.

$$f(x_0+sh) = f(x_0) + shf'(x_0) + \frac{s^2h^2}{2!}f''(x_0) + \cdots = \left\{1 + shd + \frac{s^2h^2D^2}{2!} + \cdots\right\}f(x_0) = e^{shd}f(x_0)$$

$$= \left(e^{hd}\right)^s f(x_0) = (1+\Delta)^s f(x_0)\quad (***)$$

The formula is the Newton forward difference interpolation formula, converge when |s|<1.

1.3.2 Newton formulas for the interpolating polynomial (Newton backward difference interpolation formula (NBDIF)):

Following similar steps as in subsection 1.3.1, with

$$s = \frac{x - x_n}{h} \tag{1.30}$$

We can obtain the following NBDIF

$$p_n(x) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \ldots + \frac{s(s+1)(s+2)\ldots(s+n-1)}{n!}\nabla^n f(x_n) \tag{1.31}$$

And the corresponding error

$$E_n(x) = f(x) - p_n(x) = \frac{s(s+1)(s+2)\ldots(s+n)}{(n+1)!}h^{n+1}f^{(n+1)}(c) \tag{1.32}$$

Where a<c<b.

Example 1.3.2: From above example find f(2.3) and f(5.5) by using (1.31)

For f(2.3), $x_n=x_2=3$, $h=1 \Rightarrow s=\frac{x-x_n}{h}=\frac{2.3-3}{1}=-0.7$, $y_n=22$, $\nabla y_n=17$, $\nabla^2 y_n=12$.

$$p_3(x)=f(x_n)+s\nabla f(x_n)+\frac{s(s+1)}{2!}\nabla^2 f(x_n)$$

$$f(2.3)\cong p_3(2.3)=22+(-0.7)(17)+\frac{(-0.7)(-0.7+1)}{2}12=8.84.$$

Similarly for f(5.5), but $x_n=6$, s=-0.5, $y_n=205$, $\nabla y_n=89$, $\nabla^2 y_n=30$, $\nabla^3 y_n=30$, $\nabla^4 y_n=0$.

If s=-0.7:

$$\left|E_2(x)\right|=\left|f(x)-p_2(x)\right|=\left|\frac{s(s+1)(s+2)}{3!}h^3 f^{(3)}(c)\right|=\frac{\left|s(s+1)(s+2)\right|}{4!}h^3\left|f^{(3)}(c)\right|=$$

$$\frac{\left|-0.7(-0.7+1)(-0.7+2)\right|}{6}M_3=0.455\quad M_3 \text{ where } M_3=\max_{1<c<6}\left|f^{(3)}(c)\right|$$

Note: It should be recalled again that all the previous formulas (Lagrange, Newton forward or backward) are only different representations of the same unique interpolating polynomial. When the points are not equally spaced, Lagrange's formula should be used, whereas Newton's formulas should be used in the case of equally spaced points. The forward formulas are to be used when interpolating at the beginning of the table of data whereas backward formulas are more suited when interpolating near the end of the table.

1.3.4 Inverse Interpolation:

Suppose that the function $f:[a,b]\to R$ is strictly monotonic (increasing or decreasing). Then the values of $y=f(x)$ at the (n+1) interpolating points,

$$y_i=f(x_i),\qquad i=0,\ 1,\ 2,\ \ldots,\ n \tag{1.35}$$

Are all distinct and lie between $f(a)$ and $f(b)$. We can construct the interpolating polynomial $p_n(y)$ which interpolates f^{-1} at these points: that is

$$p_n(y_i)=f^{-1}(y_i)=x_i,\qquad i=0,\ 1,\ 2,\ \ldots,\ n\ . \tag{1.36}$$

This polynomial can be used to approximate $x=f^{-1}(y)$ for a given y. If in particular y=0 then we are solving numerically

$$x=p_n(0)\cong f^{-1}(0)\Leftrightarrow f(x)=0 \tag{1.37}$$

This process is called ***inverse interpolation*** and nearly reverse the roles of x and y.

Notice that, if *f* is given analytically the successive derivatives of f^{-1}, needed for the error of the inverse interpolation, can be deduced from those of *f* by implicit differentiation.

Example 1.3.4: Use inverse interpolation at x=1.41 and 1.42 to estimate the root of $y=f(x)=x^2-2$ which lies between them and then estimate the accuracy of the obtained results.

From the table

x	1.41	1.42
y	-0.0119	0.0164

We have

$$p_1(y) = \frac{y-y_1}{y_0-y_1} f^{-1}(y_0) + \frac{y-y_0}{y_1-y_0} f^{-1}(y_1)$$

And $y_0 = -0.0119$, $y_1 = 0.0164$, $y=0$

Therefore $p_1(0) = 0.5795053 \quad (1.41) + 0.4204947 \quad (1.42) = 1.414205$

We have $y = f(x), \quad x = f^{-1}(y)$ and $1.41 \le x \le 1.42, \; -0.00119 \le y \le 0.00164$.

Now $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{f'(x)}$. Hence $\frac{d^2x}{dy^2} = \frac{d}{dy}\left[\frac{dx}{dy}\right] = \frac{d}{dy}\left[\frac{1}{f'(x)}\right] = \frac{-f''(x)}{[f'(x)]^2}\frac{dx}{dy} = \frac{-f''(x)}{[f'(x)]^3}$.

Since $f(x) = x^2-2, \quad f'(x) = 2x$ and $f''(x) = 2$,

Then

$$\frac{d^2x}{dy^2} = \left(f^{-1}\right)''(y) = \frac{-2}{8x^3} = \frac{-1}{4x^3}, \text{ and then } \left|\left(f^{-1}\right)''(y)\right| \le \frac{1}{4(1.41)^3} \cong 0.0892 \quad .$$

From

$$\left|p_1(y) - f^{-1}(y)\right| = \left|\frac{(y-y_0)(y-y_1)}{2}\left(f^{-1}\right)''(c_y)\right| \text{ where } y_0 < c_y < y_1, \text{ we find}$$

$$\left|p_1(0) - f^{-1}(0)\right| \le \frac{(0.0119)(0.0164)}{2} \times 0.0892 \cong 0.9 \times 10^{-5} .$$

Therefore

$$p_1(0) = 1.414205 \quad ,$$

Is correct to at least 4-decemial places which is true since the exact solution is $\sqrt{2} = 1.414214$.