Numerical Integrations

8.1 Introduction

The integrands could be empirical functions given by certain measured values. In all these instances, we need to resort to numerical methods of integration. Throughout many engineering fields, there are countless applications for integral calculus. Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

8.2 Trapezoidal Rule of Integration

In this method, the known function values are joined by straight lines. The area enclosed by these lines between the given end points is computed to approximate the integral as shown in Figure 8.1.

$$I = \int_a^b f(x)\,dx,$$

where $f(x)$ is called the integrand, $a =$ lower limit of integration, $b =$ upper limit of integration. Integrating polynomials is simple and is based on the calculus formula.

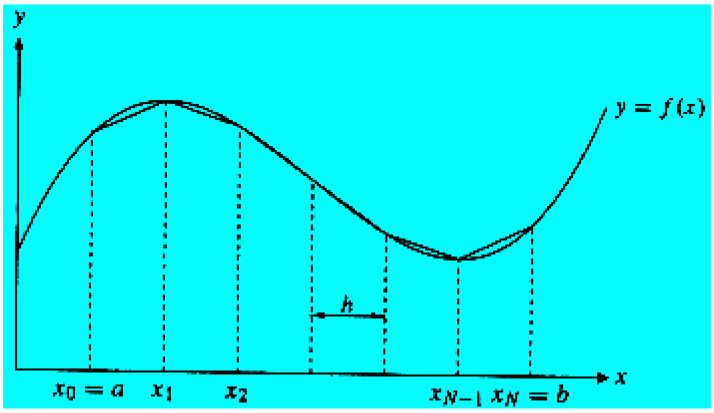


Figure 8.1 Integration of a function

$$\int_a^b x^n\,dx = \left(\frac{b^{n+1} - a^{n+1}}{n+1}\right), \; n \neq -1.$$

So if we want to approximate the integral $I = \int_a^b f(x)dx$, find the value of the above integral, one assumes $f(x) \approx f_n(x)$, where

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n,$$

is an n^{th} order polynomial. For the trapezoidal rule, assumes $n = 1$, that is, approximating the integral by a linear polynomial (straight line),

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx.$$

8.2.1 Derivation of the Trapezoidal Rule

In this section, trapezoidal rule derived by two different methods as follows:

Method 1: Derived from Calculus

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx = \int_a^b (a_0 + a_1 x)dx$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right). \qquad (8.1)$$

But what is a_0 and a_1? Now if one chooses, $(a, f(a))$ and $(b, f(b))$ as the two points to approximate $f(x)$ by a straight line from a to b,

$$f(a) = f_1(a) = a_0 + a_1 a$$

$$f(b) = f_1(b) = a_0 + a_1 b.$$

Solving the above two equations for a_1 and a_0,

$$a_1 = \frac{f(b) - f(a)}{b - a} \text{ and } a_0 = \frac{f(a)b - f(b)a}{b - a}.$$

Hence from Equation (8.1),

$$\int_a^b f(x)dx \approx \frac{f(a)b - f(b)a}{b-a}(b-a) + \frac{f(b) - f(a)}{b-a}\frac{b^2 - a^2}{2} = \frac{(b-a)}{2}\left[f(a) + f(b)\right]$$

$$= (b-a)\left[\frac{f(a) + f(b)}{2}\right] \text{ (trapezoidal rule).}$$

Method 2: Derived from Newton Forward interpolation

The trapezoidal rule can also be derived from Newton Forward interpolation. Look at Figure 8.2. The area under the curve $f_1(x)$ is the area of a trapezoid. Substituting n=1 in Equation (5.25) and considering the curve y=f(x) through the points (x_0, x_1) and (y_0, y_1) as a straight line (a polynomial of first degree so that the differences of order higher than first become zero), we get

$$I_1 = \int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} (y_0 + \Delta y_0)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= \frac{h}{2}\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1). \quad (8.2)$$

Similarly, we have

$$\left.\begin{aligned} I_2 &= \int_{x_1}^{x_2} f(x)dx = \frac{h}{2}(y_1 + y_2) \\ I_3 &= \int_{x_2}^{x_3} f(x)dx = \frac{h}{2}(y_2 + y_3). \end{aligned}\right\} \quad (8.3)$$

In general, we have

$$I_n = \int_{x_{n-1}}^{x_n} f(x)dx = \frac{h}{2}(y_{n-1} + y_n). \quad (8.4)$$

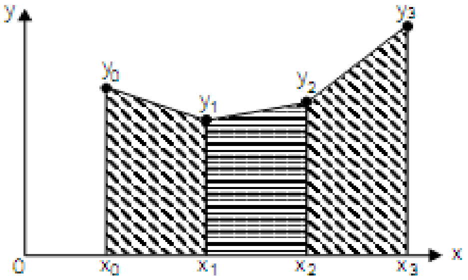


Figure 8.2 Geometric representation of trapezoidal rule.

Adding all the integrals in (8.2)-(8.4) and using the interval additive property of the definite integrals, we obtain

$$I = \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right],$$

or

$$I = \sum_{i=1}^{n} I_i = \int_{x_0}^{x_n} f(x)dx = \frac{h}{2}(y_0 + 2(y_1 + y_2 + y_3 + \ldots + y_{n-1}) + y_n) = \frac{h}{2}(E_1 + 2E_2),$$

(8.5)

where $E_1 = y_0 + y_n$ (sum of the end points), $E_2 = y_1 + y_2 + y_3 + \ldots + y_{n-1}$ (sum of the intermediate ordinates). Equation (8.5) is known as the composite trapezoidal rule.

Example 8.1: The vertical distance covered by a rocket from $t = 8$ to $t = 30$ seconds is given by $x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt.$

(a) Use the single segment trapezoidal rule to find the distance covered for $t = 8$ to $t = 30$ seconds.

(b) Find the true error, E_t for part (a).

(c) Find the absolute relative true error for part (a).

Solution: Where $a = 8$, $b = 30$

(a) $I \approx (b - a)\left[\frac{f(a) + f(b)}{2} \right]$, and let

$$f(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t .$$

$$f(8) = 2000 \ln \left[\frac{140000}{140000 - 2100(8)} \right] - 9.8(8) = 177.27 \text{ m/s.}$$

$$f(30) = 2000 \ln \left[\frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67 \text{ m/s.}$$

Hence

$$I \approx (30 - 8)\left[\frac{177.27 + 901.67}{2} \right] = 11868 \text{ m.}$$

(b) The exact value of the above integral is

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt = 11061 \text{ m.}$$

So the true error is

$E_t =$ True Value – Approximate Value

$= 11061 - 11868 = -807$ m.

(c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 = \left|\frac{11061 - 11868}{11061}\right| \times 100 = 7.2958\% .$$

Example 8.2: Use the trapezoidal rule to numerically integrate $f(x) = 0.2 + 25x$ from $a = 0$ to $b = 2$.

Solution: *f(a)* = *f*(0) = 0.2, and *f(b)* = *f*(2) = 50.2.

$$I = (b - a)\frac{f(b) + f(a)}{2} = (2 - 0)\frac{50.2 + 0.2}{2} = 50.4 .$$

The true solution is

$$I = \int_0^2 f(x)\,dx = (0.2x + 12.5x^2)\Big]_0^2 = 50.4 ,$$

because *f(x)* is a linear function, using the trapezoidal rule gets the exact solution.

Example 8.3: Use the trapezoidal rule to numerically integrate $f(x) = 0.2 + 25x + 3x^2$ from $a = 0$ to $b = 2$.

Solution: let *f(a)* = *f*(0) = 0.2, and *f(b)* = *f(2)* = 62.2.

$$I = (b - a)\frac{f(b) + f(a)}{2} = (2 - 0)\frac{622 + 0.2}{2} = 62.4 .$$

The true solution is $I = \int_0^2 f(x)\,dx = (0.2x + 12.5x^2 + x^3)\Big]_0^2 = 58.4$.

The relative error is $\left|\in_t = \left|\frac{58.4 - 62.4}{58.4}\right|\right| \times 100\% = 6.85\%$.

8.2.2 Multiple-Segment Trapezoidal Rule

In Example 8.1, the true error using a single segment trapezoidal rule was large. We can divide the interval $[8,30]$ into $[8,19]$ and $[19,30]$ intervals and apply the trapezoidal rule over each segment.

$$f(t) = 2000\ln\left(\frac{140000}{140000 - 2100t}\right) - 9.8t,$$

$$\int_{8}^{30} f(t)dt = \int_{8}^{19} f(t)dt + \int_{19}^{30} f(t)dt$$

$$\approx (19-8)\left[\frac{f(8) + f(19)}{2}\right] + (30-19)\left[\frac{f(19) + f(30)}{2}\right],$$

$f(8) = 177.27$ m/s,

$f(19) = 2000\ln\left(\frac{140000}{140000 - 2100(19)}\right) - 9.8(19) = 484.75$ m/s,

and $f(30) = 901.67$ m/s.

Hence

$$\int_{8}^{30} f(t)dt \approx (19-8)\left[\frac{177.27 + 484.75}{2}\right] + (30-19)\left[\frac{484.75 + 901.67}{2}\right] = 11266 \text{ m.}$$

The true error, E_t is $E_t = 11061 - 11266 = -205$ m.

The true error now is reduced from 807 m to 205 m. Extending this procedure to dividing $[a, b]$ into n equal segments and applying the trapezoidal rule over each segment, the sum of the results obtained for each segment is the approximate value of the integral.

Example 8.4: The vertical distance covered by a rocket from $t = 8$ to $t = 30$ seconds is given by $x = \int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right)dt$.

(a) Use the two-segment trapezoidal rule to find the distance covered from $t = 8$ to $t = 30$ seconds.

(b) Find the true error, E_t for part (a).

(c) Find the absolute relative true error for part (a).

Solution:

(a) The solution using 2-segment Trapezoidal rule is

$$I \approx \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right].$$

$$n = 2,\ a = 8,\ b = 30,\ h = \frac{b-a}{n} = \frac{30-8}{2} = 11 .$$

Hence

$$I \approx \frac{30-8}{2(2)}\left[f(8) + 2\left\{\sum_{i=1}^{2-1} f(8+11i)\right\} + f(30)\right] = \frac{22}{4}\left[f(8) + 2f(19) + f(30)\right]$$

$$= \frac{22}{4}\left[177.27 + 2(484.75) + 901.67\right] = 11266 \text{ m.}$$

(b) The exact value of the above integral is founded in Example 8.1 (b).

So the true error is

$$E_t = \text{True Value} - \text{Approximate Value} = 11061 - 11266 = -205 \text{ m} .$$

(c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 = \left|\frac{11061-11266}{11061}\right| \times 100 = 1.8537\% .$$

For other values of n, see Table 8.1.

Table 8.1 Values obtained using multiple-segment trapezoidal rule for

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right)dt .$$

n	Approximate Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Example 8.5: Use the multiple-segment trapezoidal rule to find the area under the curve $f(x) = \dfrac{300x}{1+e^x}$ from $x = 0$ to $x = 10$.

Solution: Using two segments, we get

$$h = \frac{10-0}{2} = 5,$$

$$f(0) = \frac{300(0)}{1+e^0} = 0, \quad f(5) = \frac{300(5)}{1+e^5} = 10.039,$$

$$f(10) = \frac{300(10)}{1+e^{10}} = 0.136,$$

$$I \approx \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right] = \frac{10-0}{2(2)}\left[f(0) + 2\left\{\sum_{i=1}^{2-1} f(0+5)\right\} + f(10)\right]$$

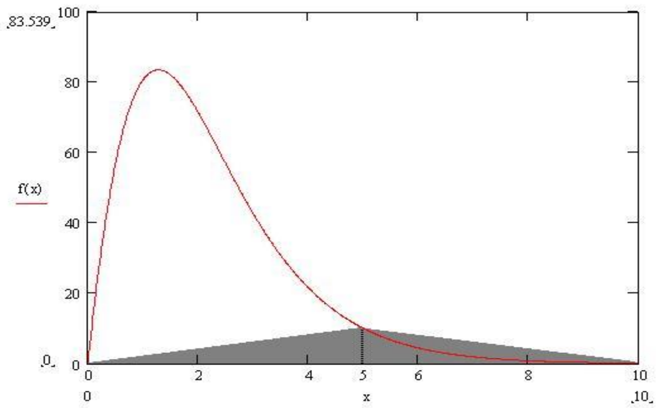
$$= \frac{10}{4}\left[f(0) + 2f(5) + f(10)\right] = \frac{10}{4}\left[0 + 2(10.039) + 0.136\right] = 50.537.$$

So what is the true value of this integral?

$$\int_0^{10} \frac{300x}{1+e^x}\,dx = 246.59.$$

Making the absolute relative true error $\left|\in_t\right| = \left|\dfrac{246.59 - 50.535}{246.59}\right| \times 100 \;\; = 79.506\%$.

Why is the true value so far away from the approximate values? Just take a look at Figure 8.3. As you can see, the area under the "trapezoids" (yeah, they really look like triangles now) covers a small portion of the area under the curve. As we add more segments, the approximated value quickly approaches the true value.



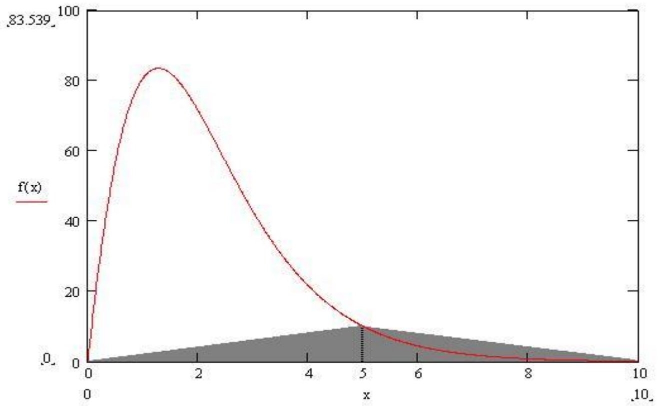


Figure 8.3 2-segment trapezoidal rule approximations.

Table 8.2 Values obtained using multiple-segment trapezoidal rule for $\int_0^{10} \frac{300x}{1+e^x} dx$.

n	Approximate Value	E_t	$\lvert \in_t \rvert$
1	0.681	245.91	99.724%
2	50.535	196.05	79.505%
4	170.61	75.978	30.812%
8	227.04	19.546	7.927%
16	241.70	4.887	1.982%
32	245.37	1.222	0.495%
64	246.28	0.305	0.124%

Example 8.6: Use multiple-segment trapezoidal rule to find $I = \int_0^2 \frac{1}{\sqrt{x}} dx$.

Solution: (H.W.)

8.2.3 Error in Multiple-segment Trapezoidal Rule

The true error for a single segment Trapezoidal rule is given by $E_t = -\frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$, where ζ is some point in $[a, b]$.

What is the error then in the multiple-segment trapezoidal rule? It will be simply the sum of the errors from each segment, where the error in each segment is that of the single segment trapezoidal rule. The error in each segment is

$$E_1 = -\frac{\left[(a+h)-a\right]^3}{12} f''(\zeta_1) = -\frac{h^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h .$$

$$E_2 = -\frac{\left[(a+2h)-(a+h)\right]^3}{12} f''(\zeta_2) = -\frac{h^3}{12} f''(\zeta_2), \quad a+h < \zeta_2 < a+2h.$$

$$\vdots$$

$$E_i = -\frac{\left[(a+ih)-(a+(i-1)h)\right]^3}{12} f''(\zeta_i) = -\frac{h^3}{12} f''(\zeta_i), \quad a+(i-1)h < \zeta_i < a+ih.$$

$$\vdots$$

$$E_n = -\frac{\left[b-\{a+(n-1)h\}\right]^3}{12} f''(\zeta_n) = -\frac{h^3}{12} f''(\zeta_n), \quad a+(n-1)h < \zeta_n < b.$$

Hence the total error in the multiple-segment trapezoidal rule is

$$E_t = \sum_{i=1}^{n} E_i = -\frac{h^3}{12}\sum_{i=1}^{n} f''(\zeta_i) = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\zeta_i) = -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}.$$

The term $\dfrac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$ is an approximate average value of the second derivative $f''(x), \ a < x < b$.

Hence $E_t = -\dfrac{(b-a)^3}{12n^2}\dfrac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$.

In Table 8.4, the approximate value of the integral

$$\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt ,$$

is given as a function of the number of segments. You can visualize that as the number of segments are doubled, the true error gets approximately quartered.

Table 8.4 Values obtained using multiple-segment trapezoidal rule for

$$x = \int_{8}^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt \, .$$

n	Approximate Value	E_t	$\lvert \in_t \rvert \%$	$\lvert \in_a \rvert \%$
2	11266	-205	1.853	5.343
4	11113	-52	0.4701	0.3594
8	11074	-13	0.1175	0.03560
16	11065	-4	0.03616	0.00401

For example, for the 2-segment trapezoidal rule, the true error is -205, and a quarter of that error is -51.25. That is close to the true error of -48 for the 4-segment trapezoidal rule.

8.3 Simpson's 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

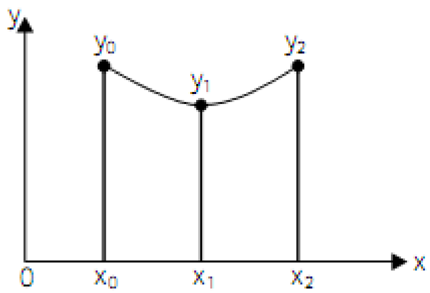


Figure 8.3 Simpson's Integration of a function

Method1: In Simpson's 1/3 rule, the function is approximated by a second degree polynomial between successive points.

Since a second degree polynomial contains three constants, it is necessary to know three consecutive function values forming two intervals as shown in Figure 8.3.

Consider three equally spaced points $x_0,\ x_1$ and x_2. Since the data are equally spaced, from the general formula of Newton-Cotes closed quadrature, let $h = x_{n+1} - x_n$,

$$I = \int_{x_0}^{x_n} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \dots \right].$$

(8.6)

Substituting $n = 2$ in Eq. (8.6) and taking the curve through the points $(x_0,\ y_0),\ (x_1, y_1)$ and (x_2, y_2) as a polynomial of second degree (parabola) so that the differences of order higher than two vanish, we obtain

$$I_1 = \int_{x_0}^{x_2} f(x)dx = 2h\left[y_0 + 4\Delta y_0 + \frac{1}{6}\Delta^2 y_0 \right] = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly, we have

$$I_2 = \int_{x_2}^{x_4} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4).$$

$$I_3 = \int_{x_4}^{x_6} f(x)dx = \frac{h}{3}(y_4 + 4y_5 + y_6).$$

In general, we have

$$I_n = \int_{x_{2n-2}}^{x_{2n}} f(x)dx = \frac{h}{3}(y_{2n-2} + 4y_{2n-1} + y_{2n}). \qquad (8.7)$$

Summing up all the above integrals, we obtain

$$I = \sum_{i=1}^{n} I_i = \int_{x_0}^{x_n} f(x)dx = \frac{h}{3}(y_0 + 4(y_1 + y_3 + \dots + y_{2n-1})$$

$$+2(y_2 + y_4 + \dots + y_{2n-2}) + y_{2n}) = \frac{h}{3}(O_1 + 4O_2 + 2O_3),$$

where $O_1 = y_0 + y_n$ (sum of end ordinates), $O_2 = y_1 + y_3 + \dots + y_{2n-1}$ (sum of odd ordinates), $O_3 = y_2 + y_4 + \dots + y_{2n-2}$ (sum of even ordinates). Equation (8.7) is known as Simpson's 1/3 rule.

Or

$$\int_a^b f(x)dx \cong \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

Simpson's 1/3 rule requires the whole range (the given interval) must be divided into even number of equal subintervals.

Example 8.7: The distance covered by a rocket in meters from $t = 8$ s to $t = 30$ s is given by $x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right)dt$.

(a) Use Simpson's 1/3 rule to find the approximate value of x.

(b) Find the true error, E_t.

(c) Find the absolute relative true error, $|\in_t|$.

Solution:

(a) $x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right], \quad a = 8, \quad b = 30, \frac{a+b}{2} = 19,$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t,$$

$$f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8) = 177.27\, m/s,$$

$$f(30) = 2000\ln\left[\frac{140000}{140000-2100(30)}\right] - 9.8(30) = 901.67\, m/s,$$

$$f(19) = 2000\ln\left[\frac{140000}{140000-2100(19)}\right] - 9.8(19) = 484.75\, m/s.$$

Hence

$$x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \left(\frac{30-8}{6}\right)\left[f(8) + 4f(19) + f(30)\right]$$

$$= \frac{22}{6}\left[177.27 + 4\times 484.75 + 901.67\right] = 11065.72 \text{ m.}$$

(b) The exact value of the above integral is given in Example 8.1. So the true error is

E_t = True vale-Approximate value =11061.34-11065.72=-4.38*m*.

(c) Absolute Relative true error, $|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 = \left|\frac{-4.38}{11061.34}\right| \times 100 = 0.0396\%$.

Example 8.8: Use 4-segment Simpson's 1/3 rule to approximate the distance covered by a rocket in meters from $t = 8$ s to $t = 30$ s as given by $x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right)dt.$

(a) Use four segment Simpson's 1/3 rule to find the *x*.

(b) Find the true error, E_t for part (a).

(c) Find the absolute relative true error, $|\in_t|$ for part (a).

Solution:

(a) Using n segment Simpson's 1/3 rule, we have

$$x \approx \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(t_i) + f(t_n)\right],$$

$$n = 4,\ a = 8,\ b = 30,\ h = \frac{b-a}{n} = \frac{30-8}{4} = 5.5$$

and

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t \cdot$$

So

$$f(t_0) = f(8) = 2000\ln\left[\frac{140000}{140000-2100(8)}\right] - 9.8(8) = 177.27\,m/s,$$

$$f(t_1) = f(8+5.5) = f(13.5) = 320.25\,m/s,$$

$$f(t_2) = f(13.5+5.5) = f(19) = 484.75\,m/s,$$

$$f(t_3) = f(19+5.5) = f(24.5) = 676.05\,m/s,$$

And $f(t_4) = f(24.5+5.5) = f(30) = 901.67 m/s.$

Hence

$$x \approx \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(t_i) + f(t_n) \right],$$

$$= \frac{30-8}{3(4)}\left[f(8) + 4\sum_{\substack{i=1 \\ i=odd}}^{3} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{2} f(t_i) + f(30) \right]$$

$$= \frac{22}{12}\left[f(8) + 4f(t_1) + 4f(t_3) + 2f(t_2) + f(30) \right]$$

$$= \frac{11}{6}\left[f(8) + 4f(13.5) + 4f(24.5) + 2f(19) + f(30) \right]$$

$$= \frac{11}{6}\left[177.27 + 4(320.25) + 4(676.05) + 2(484.75) + 901.67 \right]$$

$$= 11061.64 m.$$

(b) The exact value of the above integral is given in Example 8.1. So the true error is

E_t = True vale-Approximate value =11061.34-11061.64=-0.30*m*.

(c) Absolute Relative true error, $|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100 = \left|\frac{-0.30}{11061.34}\right| \times 100 = 0.0027\%.$

Table 8.5 Values of Simpson's 1/3 rule for Example 8.6 with multiple-segments

n	Approximate Value	E_t	$\lvert \in_t \rvert$
2	11065.72	-4.38	0.0396%
4	11061.64	-0.30	0.0027%
6	11061.40	-0.06	0.0005%
8	11061.35	-0.02	0.0002%
10	11061.34	-0.01	0.0001%

Example 8.9: Use Simpson's($\frac{1}{3}$) rule to integrate

$f(x) = 0.2 + 25x + 3x^2 + 8x^3$ from $a = 0$ to $b = 2$.

Solution: let $f(0)=0.2$, $f(1)=36.2$ and $f(2)=126.2$

$$I=(b-a)\frac{f(0)+4f(1)+f(2)}{2}=(2-0)\frac{0.2+2\times 36.2+126.2}{2}=90.4$$

The exact integral is $I=\int_0^2 f(x)\,dx=(0.2x+12.5x^2+x^3+2x^4)\Big]_0^2=90.4$

Example 8.10: Use Simpson's $(\frac{1}{3})$ rule to integrate $f(x)=0.2+25x+3x^2+2x^4$ from $a=0$ to $b=2$.

Solution: let $f(0)=0.2$, $f(1)=30.2$ and $f(2)=94.2$

$$I=(b-a)\frac{f(0)+4f(1)+f(2)}{2}=(2-0)\frac{0.2+2\times 30.2+94.2}{2}=71.73$$

The exact integral is

$$I=\int_0^2 f(x)\,dx=(0.2x+12.5x^2+x^3+0.4x^5)\Big]_0^2=71.2 .$$

The relative error is $\left|\in_t=\left|\frac{71.2-71.73}{71.2}\right|\right|\times 100\%=0.7\%$.

8.3.2 Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3rd Rule is given by

$$E_t=-\frac{(b-a)^5}{2880}f^{(4)}(\zeta),\ a<\zeta<b.$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the *n* segments Simpson's 1/3 rule is given by

$$E_1=-\frac{(x_2-x_0)^5}{2880}f^{(4)}(\zeta_1)=-\frac{h^5}{90}f^{(4)}(\zeta_1),\quad x_0<\zeta_1<x_2.$$

$$E_2=-\frac{(x_4-x_2)^5}{2880}f^{(4)}(\zeta_2)=-\frac{h^5}{90}f^{(4)}(\zeta_2),\quad x_2<\zeta_2<x_4.$$

$\vdots$

$$E_i=-\frac{(x_{2i}-x_{2(i-1)})^5}{2880}f^{(4)}(\zeta_i)=-\frac{h^5}{90}f^{(4)}(\zeta_i),\quad x_{2(i-1)}<\zeta_i<x_{2i}.$$

$\vdots$

$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}(\zeta_{\frac{n}{2}-1}) = -\frac{h^5}{90} f^{(4)}(\zeta_{\frac{n}{2}-1}), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}. \text{ and}$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right) = -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), x_{n-2} < \zeta_{\frac{n}{2}} < x_n .$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i = -\frac{h^5}{90}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) = -\frac{(b-a)^5}{90n^5}\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i) = -\frac{(b-a)^5}{90n^4}\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}.$$

The term $\frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of $f^{(4)}(x),\ .\ a < x < b$.

Hence $E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)},$ (8.14)

where $\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$.

8.4 Simpson 3/8 Rule for Integration

In a similar fashion, Simpson 3/8 rule for integration can be derived by approximating the given function $f(x)$ with the 3rd order (cubic) polynomial. Putting n=3 in (8.6) and taking the curve through (x_n, y_n), $n = 0, 1, 2, 3$ as a polynomial of degree three such that the differences higher than the third order vanish, we obtain

$$I_1 = \int_{x_0}^{x_3} f(x)dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{2}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3).$$

Similarly, we get

$$I_2 = \int_{x_3}^{x_6} f(x)dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)\ .$$

$$I_3 = \int_{x_6}^{x_9} f(x)dx = \frac{3h}{8}(y_6 + 3y_7 + 3y_8 + y_9)\ .$$

In general, we have

$$I_n = \int_{x_{3n-3}}^{x_{3n}} f(x)dx = \frac{3h}{8}(y_{3n-3} + 3y_{3n-2} + 3y_{3n-1} + y_{3n}).$$

Summing up all the above integrals, we obtain

$$I = \int_{x_0}^{x_n} f(x)dx = \frac{h}{3}[(y_0 + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 \ldots + y_{3n-2} + y_{3n-1})$$

$$+2(y_3 + y_6 + y_9 + \ldots + y_{3n-3}) + y_{3n}], \qquad (8.9)$$

or

$$I = \frac{3h}{8}\left\{ f(x_0) + 3\sum_{i=1,4,7,..}^{n-2} f(x_i) + 3\sum_{i=2,5,8,..}^{n-1} f(x_i) + 2\sum_{i=3,6,9,..}^{n-3} f(x_i) + f(x_n)\right\}.$$

Equation (8.9) is called the Simpson's 3/8 rule. Here, the number of subintervals should be taken as multiples of 3. Simpson's 3/8 rule is not as accurate as Simpson's 1/3 rule.

The true error in Simpson 3/8 rule can be derived as

$$E_t = -\frac{(b-a)^5}{6480} \times f''''(\zeta)\ , \text{ where } a \le \zeta \le b. \qquad (8.10)$$

Example 8.11: The vertical distance covered by a rocket from $x = 8$ to $x = 30$ seconds is given by $s = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8x\right)dx$.

Use Simpson 3/8 rule to find the approximate value of the integral.

Solution: $h = \frac{b-a}{n} = \frac{b-a}{3} = \frac{30-8}{3} = 7.3333.$ Hence

$$I \approx \frac{3h}{8} \times \{ f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}\ .$$

$$f\left(x_0\right) = f(8) = 2000\ln\left(\frac{140000}{140000-2100\times 8}\right) - 9.8\times 8 = 177.2667 \ m/s.$$

$$f(x_1) = f(x_0+h) = f(15.3333) - 372.4629 m/s.$$

$$f\left(x_2\right) = f(x_0+2h) = f(22.6666) = 608.8976 m/s.$$

$$f\left(x_3\right) = f(x_0+3h) = f(30) = 901.6740 m/s.$$

Applying Equation (8.9), one has

$$I = \frac{3}{8}\times 7.3333\times\{177.2667+3\times 372.4629+3\times 608.8976+901.6740\}$$

$= 11063.3104.$

The exact answer can be computed as

$$I_{exact} = 11061.34 .$$

Example 8.12: The vertical distance covered by a rocket from $x=8$ to $x=30$ seconds is given by $s=\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8x\right)dx$.

Use Simpson 3/8 multiple segments rule with six segments to estimate the vertical distance.

Solution: $h=\frac{30-8}{6}=3.6666.$

$$\{x_0, f\left(x_0\right)\} = \{8, 177.2667\} .$$

$\{x_1, f\left(x_1\right)\} = \{11.6666, 270.4104\}$, where

$$x_1 = x_0 + h = 8 + 3.6666 = 11.6666 .$$

$$\{x_2, f\left(x_2\right)\} = \{15.3333, 372.4629\} \text{ where } x_2 = x_0 + 2h = 15.3333.$$

$$\{x_3, f\left(x_3\right)\} = \{19, 484.7455\} \text{ where } x_3 = x_0 + 3h = 19.$$

$$\{x_4, f\left(x_4\right)\} = \{22.6666, 608.8976\} \text{ where } x_4 = x_0 + 4h = 22.6666.$$

$$\{x_5, f\left(x_5\right)\} = \{26.3333, 746.9870\} \text{ where } x_5 = x_0 + 5h = 26.3333.$$

$$\{x_6, f\left(x_6\right)\} = \{30, 901.6740\} \text{ where } x_6 = x_0 + 6h = 30.$$

Applying Equation (8.9), one obtains:

$$I = \frac{3}{8}(3.6666)\left\{177.2667 + 3\sum_{i=1,4,..}^{n-2=4} f(x_i) + 3\sum_{i=2,5,..}^{n-1=5} f(x_i) + 2\sum_{i=3,6,..}^{n-3=3} f(x_i) + 901.6740\right\}$$

$$= (1.3750)\{177.2667 + 3(270.4104 + 608.8976) + 3(372.4629 + 746.9870)$$

$$+2(484.7455) + 901.6740\} = 11,601.4696.$$

Example 8.13: Compute $I = \int_{a=8}^{b=30} \left\{2000\ln\left(\frac{140,000}{140,000 - 2100x}\right) - 9.8x\right\} dx,$

using Simpson 1/3 rule (with $n_1 = 4$), and Simpson 3/8 rule (with $n_2 = 3$).

Solution: The segment width is $h = \frac{b-a}{n} = \frac{b-a}{n_1 + n_2} = \frac{30-8}{(4+3)} = 3.1429.$

$$\left.\begin{aligned} x_0 &= a = 8 \\ x_1 &= x_0 + 1h = 8 + 3.1429 = 11.1429 \\ x_2 &= x_0 + 2h = 8 + 2(3.1429) = 14.2857 \\ x_3 &= x_0 + 3h = 8 + 3(3.1429) = 17.4286 \\ x_4 &= x_0 + 4h = 8 + 4(3.1429) = 20.5714 \end{aligned}\right\} \text{Simpson's 1/3 rule}$$

$$x_5 = x_0 + 5h = 8 + 5(3.1429) = 23.7143$$

$$x_6 = x_0 + 6h = 8 + 6(3.1429) = 26.8571$$

$$x_7 = x_0 + 7h = 8 + 7(3.1429) = 30,$$

$$f(x_0) = f(8) = 2000\ln\left(\frac{140,000}{140,000 - 2100\times 8}\right) - 9.8\times 8 = 177.2667.$$

Similarly:

$$f(x_1 = 11.1429) = 256.5863, \quad f(x_2) = 342.3241$$

$$f(x_3) = 435.2749, \quad f(x_4) = 536.3909, \quad f(x_5) = 646.8260,$$

$$f(x_6) = 767.9978, \quad f(x_7) = 901.6740.$$

For multiple segments $(n_1 = \text{first 4 segments})$, using Simpson 1/3 rule, one obtains (See Equation (8.9)):

$$I_1 = \left(\frac{h}{3}\right)\left\{ f(x_0) + 4\sum_{i=1,3,\ldots}^{n_1-1=3} f(x_i) + 2\sum_{i=2,\ldots}^{n_1-2=2} f(x_i) + f\left(x_{n_1}\right)\right\}$$

$$= \left(\frac{h}{3}\right)\left\{ f(x_0) + 4\left(f(x_1) + f(x_3)\right) + 2 f(x_2) + f(x_4)\right\}$$

$$= \left(\frac{3.1429}{3}\right)\left\{177.2667 + 4\left(256.5863 + 435.2749\right) + 2\left(342.3241\right) + 536.3909\right\}$$

$= 4364.1197$.

For multiple segments $\left(n_2 = \text{last 3 segments}\right)$, using Simpson 3/8 rule, one obtains (See Equation (8.9)):

$$I_2 = \left(\frac{3h}{8}\right)\left\{ f(t_0) + 3\sum_{i=1,3,\ldots}^{n_2-2=1} f(t_i) + 3\sum_{i=2,\ldots}^{n_2-1=2} f(t_i) + 2\sum_{i=3,6,..}^{n_2-3=0} f(t_i) + f\left(t_{n_1}\right)\right\}$$

$$= \left(\frac{3h}{8}\right)\left\{ f(t_0) + 3 f(t_1) + 3 f(t_2) + 2(\text{no contribution}) + f(t_3)\right\}$$

$$= \left(\frac{3h}{8}\right)\left\{ f(x_4) + 3 f(x_5) + 3 f(x_6) + f(x_7)\right\}$$

$$= \left(\frac{3}{8}\times 3.1429\right)\left\{536.3909 + 3\left(646.8260\right) + 3\left(767.9978\right) + 901.6740\right\}$$

$= 6697.3663.$

The mixed (combined) Simpson 1/3 and 3/8 rules give

$$I = I_1 + I_2 = 4364.1197 + 6697.3663 = 11061.$$

(i) Comparing the truncated error of Simpson 1/3 rule.

(ii) $E_t = -\dfrac{(b-a)^5}{2880}\times f''''(\zeta).$ (8.11)

With Simpson 3/8 rule (See Equation (8.8)), it seems to offer slightly more accurate answer than the former. However, the cost associated with Simpson 3/8 rule (using 3rd order polynomial function) is significantly higher than the one associated with Simpson 1/3 rule (using 2nd order polynomial function).

The number of multiple segments that can be used in the conjunction with Simpson 1/3 rule is 2, 4, 6, 8, … (any even numbers).

$$I_1 = \left(\frac{h}{3}\right)\{ f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \ldots.. + f(x_{n-2})$$

$$+4f(x_{n-1}) + f(x_n)\} = \left(\frac{h}{3}\right)\left\{ f(x_0) + 4\sum_{i=1,3,\ldots}^{n-1} f(x_i) + 2\sum_{i=2,4,6\ldots}^{n-2} f(x_i) + f(x_n)\right\}.$$

However, Simpson 3/8 rule can be used with the number of segments equal to 3, 6, 9, 12, ... (can be certain odd or even numbers that are multiples of 3). If the user wishes to use, say 7 segments, then the mixed Simpson 1/3 rule (for the first 4 segments), and Simpson 3/8 rule (for the last 3 segments) would be appropriate.

8.7 Gauss Quadrature Rule of Integration

To derive the trapezoidal rule from the method of undetermined coefficients, we approximated

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b). \qquad (8.23)$$

Let the right hand side be exact for integrals of a straight line, that is, for an integrated form of

$$\int_a^b (a_0 + a_1 x)\,dx.$$

So

$$\int_a^b (a_0 + a_1 x)\,dx = \left[a_0 x + a_1 \frac{x^2}{2} \right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right). \qquad (8.24)$$

But from Equation (8.23), we want

$$\int_a^b (a_0 + a_1 x)\,dx = c_1 f(a) + c_2 f(b).$$

To give the same result as Equation (8.24) for $f(x) = a_0 + a_1 x$,

$$\int_a^b (a_0 + a_1 x)dx = c_1(a_0 + a_1 a) + c_2(a_0 + a_1 b)$$

$$= a_0 (c_1 + c_2) + a_1 (c_1 a + c_2 b). \qquad (8.25)$$

Hence from Equations (8.24) and (8.25),

$$a_0 (b - a) + a_1 \left(\frac{b^2 - a^2}{2} \right) = a_0 (c_1 + c_2) + a_1 (c_1 a + c_2 b).$$

Since a_0 and a_1 are arbitrary constants for a general straight line

$$c_1 + c_2 = b - a\,, \qquad (8.26)$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2}. \qquad (8.27)$$

Multiplying Equation (8.26) by a and subtracting from Equation (8.27) gives

$$c_2 = \frac{b - a}{2}.$$

Substituting the above value of c_2 in Equation (8.26) gives $c_1 = \frac{b - a}{2}.$

Therefore $\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b) = \frac{b - a}{2} f(a) + \frac{b - a}{2} f(b).$

8.7.1 Derivation of two-point Gauss quadrature rule

Method 1: The two-point Gauss quadrature rule is an extension of the trapezoidal rule approximation where the arguments of the function are not predetermined as a and b, but as unknowns x_1 and x_2. So in the two-point Gauss quadrature rule, the integral is approximated as

$$I = \int_a^b f(x)dx$$

$$\approx c_1 f(x_1) + c_2 f(x_2).$$

There are four unknowns x_1, x_2, c_1 and c_2. These are found by assuming that the formula gives exact results for integrating a general third order polynomial, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Hence

$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0 (b - a) + a_1 \left(\frac{b^2 - a^2}{2}\right) + a_2 \left(\frac{b^3 - a^3}{3}\right) + a_3 \left(\frac{b^4 - a^4}{4}\right). \quad (8.28)$$

The formula would then give

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= c_1 \left(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3\right) + c_2 \left(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3\right). \quad (8.29)$$

Equating Equations (8.28) and (8.29) gives

$$a_0 (b - a) + a_1 \left(\frac{b^2 - a^2}{2}\right) + a_2 \left(\frac{b^3 - a^3}{3}\right) + a_3 \left(\frac{b^4 - a^4}{4}\right)$$

$$= c_1 (a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + c_2 (a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

$$= a_0 (c_1 + c_2) + a_1 (c_1 x_1 + c_2 x_2) + a_2 (c_1 x_1^2 + c_2 x_2^2) + a_3 (c_1 x_1^2 + c_2 x_2^3). \quad (8.30)$$

Since in Equation (8.30), the constants a_0, a_1, a_2, and a_3 are arbitrary, the coefficients of a_0, a_1, a_2, and a_3 are equal. This gives us four equations as follows.

$$b - a = c_1 + c_2,$$

$$\frac{b^2 - a^2}{2} = c_1 x_1 + c_2 x_2,$$

$$\frac{b^3 - a^3}{3} = c_1 x_1^2 + c_2 x_2^2,$$

$$\frac{b^4 - a^4}{4} = c_1 x_1^3 + c_2 x_2^3.$$

Without proof (see Example 8.12 for proof of a related problem), we can find that the above four simultaneous nonlinear equations have only one acceptable solution

$$c_1 = \frac{b-a}{2},\quad c_2 = \frac{b-a}{2},\quad x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2},$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}.$$

Hence

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right).$$

Method 2: We can derive the same formula by assuming that the expression gives exact values for the individual integrals of $\int_a^b 1dx,\ \int_a^b xdx,\ \int_a^b x^2dx,$ and $\int_a^b x^3dx$.

The reason the formula can also be derived using this method is that the linear combination of the above integrands is a general third order polynomial given by $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

These will give four equations as follows

$$\int_a^b 1dx = b - a = c_1 + c_2,$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1x_1 + c_2x_2,$$

$$\int_a^b x^2dx = \frac{b^3 - a^3}{3} = c_1x_1^2 + c_2x_2^2,$$

$$\int_a^b x^3dx = \frac{b^4 - a^4}{4} = c_1x_1^3 + c_2x_2^3.$$

These four simultaneous nonlinear equations can be solved to give a single acceptable solution

$$c_1 = \frac{b-a}{2},\ c_2 = \frac{b-a}{2},\quad x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2},$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}.$$

Hence

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right) + \frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right).$$

Since two points are chosen, it is called the two-point Gauss quadrature rule. Higher point versions can also be developed.

8.7.2 Higher point Gauss quadrature formulas

For example

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3), \qquad (8.53)$$

is called the three-point Gauss quadrature rule. The coefficients c_1, c_2 and c_3, and the function arguments x_1, x_2 and x_3 are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right) dx.$$

General n -point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots. + c_n f(x_n). \qquad (8.54)$$

8.7.3 Arguments and weighing factors for n-point Gauss quadrature rules

In handbooks (see Table 8.9), coefficients and arguments given for n -point Gauss quadrature rule are given for integrals of the form

$$\int_{-1}^{1} g(x)dx \approx \sum_{i=1}^{n} c_i g(x_i)\,.$$

Table 8.9 Weighting factors c and function arguments x used in Gauss quadrature formulas

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$	$x_1 = -0.577350269$
	$c_2 = 1.000000000$	$x_2 = 0.577350269$
3	$c_1 = 0.555555556$	$x_1 = -0.774596669$
	$c_2 = 0.888888889$	$x_2 = 0.000000000$
	$c_3 = 0.555555556$	$x_3 = 0.774596669$
4	$c_1 = 0.347854845$	$x_1 = -0.861136312$
	$c_2 = 0.652145155$	$x_2 = -0.339981044$
	$c_3 = 0.652145155$	$x_3 = 0.339981044$
	$c_4 = 0.347854845$	$x_4 = 0.861136312$
5	$c_1 = 0.236926885$	$x_1 = -0.906179846$
	$c_2 = 0.478628670$	$x_2 = -0.538469310$
	$c_3 = 0.568888889$	$x_3 = 0.000000000$
	$c_4 = 0.478628670$	$x_4 = 0.538469310$
	$c_5 = 0.236926885$	$x_5 = 0.906179846$
6	$c_1 = 0.171324492$	$x_1 = -0.932469514$
	$c_2 = 0.360761573$	$x_2 = -0.661209386$
	$c_3 = 0.467913935$	$x_3 = -0.238619186$
	$c_4 = 0.467913935$	$x_4 = 0.238619186$
	$c_5 = 0.360761573$	$x_5 = 0.661209386$
	$c_6 = 0.171324492$	$x_6 = 0.932469514$

So if the table is given for $\int_{-1}^{1} g(x)dx$ integrals, how does one solve $\int_{a}^{b} f(x)dx$?

The answer lies in that any integral with limits of $[a, b]$ can be converted into an integral with limits $[-1, 1]$. Let

$$x = mt + c .$$

If $x = a$, then $t = -1$

If $x = b$, then $t = +1$

such that

$$\left.\begin{aligned} a &= m(-1) + c \\ b &= m(1) + c. \end{aligned}\right\} \qquad (8.31)$$

Solving the two Equations (8.31) simultaneously gives

$$m = \frac{b-a}{2} \text{ and } c = \frac{b+a}{2}.$$

Hence

$$x = \frac{b-a}{2}t + \frac{b+a}{2} \text{ and } dx = \frac{b-a}{2}dt.$$

Substituting our values of x and dx into the integral gives us

$$\int_{a}^{b} f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)\frac{b-a}{2}dx. \qquad (8.32)$$

Example 8.17: For an integral $\int_{a}^{b} f(x)dx$, derive the one-point Gauss quadrature rule.

Solution: The one-point Gauss quadrature rule is

$$\int_{a}^{b} f(x)dx \approx c_1 f(x_1).$$

Assuming the formula gives exact values for integrals $\int_{-1}^{1} 1dx$, and $\int_{-1}^{1} xdx$.

$$\int_a^b 1dx = b - a = c_1,$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 x_1.$$

Since $c_1 = b - a$, the other equation becomes $(b-a)x_1 = \frac{b^2 - a^2}{2}$, $x_1 = \frac{b+a}{2}$.

Therefore, one-point Gauss quadrature rule can be expressed as

$$\int_a^b f(x)dx \approx (b-a) f\left(\frac{b+a}{2}\right).$$

Example 8.18: What would be the formula for $\int_a^b f(x)dx = c_1 f(a) + c_2 f(b)$,

if you want the above formula to give you exact values of $\int_a^b \left(a_0 x + b_0 x^2\right) dx$, that is, a linear combination of x and x^2.

Solution: If the formula is exact for a linear combination of x and x^2, then

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 a + c_2 b,$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 b^2.$$

The above equations, in matrix form can be written as follows:

$$\begin{bmatrix} a & b \\ a^2 & b^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{b^2 - a^2}{2} \\ \frac{b^3 - a^3}{3} \end{bmatrix}.$$

Solving the two above equations simultaneously gives

$$c_1 = -\frac{1}{6}\frac{-ab - b^2 + 2a^2}{a}, \quad c_2 = -\frac{1}{6}\frac{a^2 + ab - 2b^2}{b}.$$

So

$$\int_a^b f(x)dx = -\frac{1}{6}\frac{-ab-b^2+2a^2}{a}f(a) - \frac{1}{6}\frac{a^2+ab-2b^2}{b}f(b). \quad (8.43)$$

Let us see if the formula works.

Evaluate $\int_2^5 \left(2x^2 - 3x\right)dx$ using Equation (8.43):

$$\int_2^5 \left(2x^2 - 3x\right)dx \approx c_1 f(a) + c_2 f(b) = -\frac{1}{6}\frac{-(2)(5) - 5^2 + 2(2)^2}{2}\left[2(2)^2 - 3(2)\right]$$

$$-\frac{1}{6}\frac{2^2 + 2(5) - 2(5)^2}{5}[2(5)^2 - 3(5)] = 46.5 .$$

The exact value of $\int_2^5 \left(2x^2 - 3x\right)dx$ is given by

$$\int_2^5 \left(2x^2 - 3x\right)dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2}\right]_2^5 = 46.5 .$$

Any surprises?

Now evaluate $\int_2^5 3dx$ using Equation (8.43)

$$\int_2^5 3dx \approx c_1 f(a) + c_2 f(b)$$

$$= -\frac{1}{6}\frac{-2(5) - 5^2 + 2(2)^2}{2}(3) - \frac{1}{6}\frac{2^2 + 2(5) - 2(5)^2}{5}(3) = 10.35.$$

The exact value of $\int_2^5 3dx$ is given by

$$\int_2^5 3dx = [3x]_2^5 = 9.$$

Because the formula will only give exact values for linear combinations of x and x^2, it does not work exactly even for a simple integral of $\int_2^5 3dx$.

Do you see now why we choose $a_0 + a_1 x$ as the integrand for which the formula

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b),$$

gives us exact values?

Example 8.19: Use two-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt .$$

Also, find the absolute relative true error.

Solution: First, change the limits of integration from $[8, 30]$ to $[-1, 1]$ using Equation (8.32) gives

$$\int_8^{30} f(t)dt = \frac{30-8}{2} \int_{-1}^{1} f\left(\frac{30-8}{2} x + \frac{30+8}{2} \right) dx = 11 \int_{-1}^{1} f(11x + 19) dx.$$

Next, get weighting factors and function argument values from Table 8.9 for the two point rule,

$c_1 = 1.000000000$, $x_1 = -0.577350269$, $c_2 = 1.000000000$ and

$x_2 = 0.577350269$.

Now we can use the Gauss quadrature formula

$$11 \int_{-1}^{1} f(11x + 19) dx \approx 11 \left[c_1 f(11x_1 + 19) + c_2 f(11x_2 + 19) \right]$$

$$= 11 \left[f(11(-0.5773503) + 19) + f(11(0.5773503) + 19) \right]$$

$$= 11 \left[f(12.64915) + f(25.35085) \right]$$

$$= 11 \left[(296.8317) + (708.4811) \right] = 11058.44 \text{ m}.$$

Since

$$f(12.64915) = 2000 \ln \left[\frac{140000}{140000 - 2100(12.64915)} \right] - 9.8(12.64915) = 296.8317,$$

$$f(25.35085) = 2000 \ln \left[\frac{140000}{140000 - 2100(25.35085)} \right] - 9.8(25.35085) = 708.4811.$$

The absolute relative true error, $|\in_t|$, is (True value = 11061.34 m)

$$|\in_t| = \left|\frac{11061.34 - 11058.44}{11061.34}\right| \times 100 = 0.0262\% \ .$$

Example 8.20: Solve Example 8.15 by using three-point Gauss quadrature rule.

Solution: First, change the limits of integration from $[8, 30]$ to $[-1, 1]$ using Equation (8.32) gives

$$\int_8^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2}\right)dx \ = 11\int_{-1}^{1} f\left(11x+19\right)dx.$$

The weighting factors and function argument values are

$$c_1 = \ 0.555555556\ ,\ x_1 = -0.774596669\ ,\ c_2 = \ 0.888888889$$

$$x_2 = \ 0.000000000\ ,\ c_3 = \ 0.555555556 \text{ and } x_3 = \ 0.774596669\ ,$$

and the formula is

$$11\int_{-1}^{1} f\left(11x+19\right)dx \approx 11\left[c_1 f\left(11x_1+19\right) + c_2 f\left(11x_2+19\right) + c_3 f\left(11x_3+19\right)\right]$$

$$= 11[0.5555556\, f\left(11(-.7745967)+19\right) + 0.8888889\, f\left(11(0.0000000)+19\right)$$

$$+0.5555556\, f\left(11(0.7745967)+19\right)]$$

$$= 11\left[0.55556\, f(10.47944) + 0.88889\, f(19.00000) + 0.55556\, f(27.52056)\right]$$

$$= 11\left[0.55556 \times 239.3327 + 0.88889 \times 484.7455 + 0.55556 \times 795.1069\right]$$

$$= 11061.31 \text{ m} \ .$$

Since

$$f(10.47944) = 2000\ln\left[\frac{140000}{140000 - 2100(10.47944)}\right] - 9.8(10.47944) = 239.3327\ ,$$

$$f(19.00000) = 2000\ln\left[\frac{140000}{140000 - 2100(19.00000)}\right] - 9.8(19.00000) = 484.7455\ ,$$

$$f(27.52056) = 2000\ln\left[\frac{140000}{140000 - 2100(27.52056)}\right] - 9.8(27.52056) = 795.1069\ .$$

The absolute relative true error, $\left|\in_t\right|$, is (True value = 11061.34 m)

$$\left|\in_t\right| = \left|\frac{11061.34 - 11061.31}{11061.34}\right| \times 100 \ = 0.0003\%$$

8.8 Gauss-Legendre Integration Methods

Consider the formula

$$\int_{-1}^{1} f(x)dx = \sum_{k=0}^{n} \alpha_k f(x_k).$$

In this case, all the nodes x_k and weight α_k are unknown. Consider the following cases.

One-point formula: let $n = 0$. The formula is given by

$$\int_{-1}^{1} f(x)dx = \alpha_0 f(x_0).$$

The method has two unknowns α_0, x_0. Making the exact for $f(x) = 1, x$, we get

$$f(x) = 1: \ 2 = \alpha_0$$

$$f(x) = x: \ 0 = \alpha_0 x_0 \ \text{or} \ x_0 = 0.$$

Hence, the method is given by $\int_{-1}^{1} f(x)dx = 2f(0).$

The error constants is given by $C = \int_{-1}^{1} x^2 dx - 2[0] = \frac{2}{3}.$

$$\text{Hence } R_1 = \frac{C}{2!} f''(\lambda) = \frac{1}{3} f''(\lambda), \ \ -1 < \lambda < 1.$$

Two-point formula: Let $n = 1$. The formula is given by

$$\int_{-1}^{1} f(x)dx = \alpha_0 f(x_0) + \alpha_1 f(x_1).$$

The method has two unknowns α_0, x_0, α_1 and x_1. Making the exact for $f(x) = 1, x, x^2, x^3$, we get

$$f(x)=1:\ 2=\alpha_0+\alpha_1,$$
$$f(x)=x:\ 0=\alpha_0x_0+\alpha_1x_1,$$
$$f(x)=x^2:\ \frac{2}{3}=\alpha_0x_0^2+\alpha_1x_1^2,$$
$$f(x)=x^3:\ 0=\alpha_0x_0^3+\alpha_1x_1^3.$$

Eliminating for α_0, we get $\alpha_0=\alpha_1=1$.

And also $x_0=\pm\frac{1}{\sqrt{3}}$ and $x_1=\pm\frac{1}{\sqrt{3}}$.

There for, the two point Gauss Legendre method is given by

$$\int_{-1}^{1} f(x)dx = f(-\frac{1}{\sqrt{3}})+f(\frac{1}{\sqrt{3}}).$$

The error constants is given by $C=\int_{-1}^{1} x^4dx-[\frac{1}{9}+\frac{1}{9}]=\frac{8}{45}.$

Hence $R_4=\frac{C}{4!}f^{(4)}(\lambda)=\frac{1}{135}f^{(4)}(\lambda),\ -1<\lambda<1.$

Three-point formula: Let $n=2$. The formula is given by

$$\int_{-1}^{1} f(x)dx=\alpha_0f(x_0)+\alpha_1f(x_1)+\alpha_2f(x_2).$$

The method has two unknowns $\alpha_0,\ x_0,\ \alpha_1,\ x_1,\alpha_2$ and x_2. Making the exact for $f(x)=1,\ x,\ x^2,\ x^3,x^4,x^5$, we get

$$f(x)=1:\quad 2=\alpha_0+\alpha_1+\alpha_2,$$
$$f(x)=x:\quad 0=\alpha_0x_0+\alpha_1x_1+\alpha_2x_2,$$
$$f(x)=x^2:\quad \frac{2}{3}=\alpha_0x_0^2+\alpha_1x_1^2+\alpha_2x_2^2,$$
$$f(x)=x^3:\quad 0=\alpha_0x_0^3+\alpha_1x_1^3+\alpha_2x_3^3,$$
$$f(x)=x^4:\quad \frac{2}{5}=\alpha_0x_0^4+\alpha_1x_1^4+\alpha_2x_3^4,$$
$$f(x)=x^5:\quad 0=\alpha_0x_0^5+\alpha_1x_1^5+\alpha_2x_3^5.$$

Eliminating for α_0, we get $\alpha_0=\frac{5}{9}=\alpha_2$ and $\alpha_1=\frac{8}{9}.$

And also $x_0 = \pm\sqrt{\frac{3}{5}}$, $x_1 = 0$ and $x_2 = \mp\sqrt{\frac{3}{5}}$.

There for, the three point Gauss Legendre method is given by

$$\int_{-1}^{1} f(x)dx = \frac{1}{9}\left[5f(-\sqrt{\frac{3}{5}}) + 8f(0) + 5f(\sqrt{\frac{3}{5}})\right].$$

The error constants is given by

$$C = \int_{-1}^{1} x^6 dx - \frac{1}{9}[5(-\sqrt{\frac{3}{5}})^6 + 0 + 5(\sqrt{\frac{3}{5}})^6] = \frac{8}{175}.$$

Hence

$$R_6 = \frac{C}{6!} f^{(6)}(\lambda) = \frac{1}{(6!)175} f^{(6)}(\lambda) = \frac{1}{15750} f^{(6)}(\lambda), \quad -1 < \lambda < 1.$$

Example 8.21: Evaluate the integral $\int_{1}^{2} \frac{2x}{1+x^4} dx$, using Gauss-Legendre 1-point, 2-point and 3-point quadrature rules. Compare with the exact solution $I = \tan^{-1}(4) - \frac{\pi}{4}$.

Solution: first change the interval $[1, 2]$ to $[-1, 1]$. Writing $x = at + b$, we get

$$1 = -a + b \;,\; 2 = a + b,$$

whose solution is $b = \frac{3}{2}, \; a = \frac{1}{2}$. There for, $x = (t+3)/2, \; dx = dt/2,$ and

$$\int_{1}^{2} \frac{2x}{1+x^4} dx = \int_{-1}^{1} \frac{8(t+3)}{[16+(t+3)^4]} dt = \int_{-1}^{1} f(t)dt.$$

Using 1-point rule, we get

$$I = 2f(0) = 2\left[\frac{24}{16+18}\right] = 0.4948.$$

Using 2-point rule, we get

$$I = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}}) = 0.3842 + 0.1592 = 0.5434.$$

Using 3-point rule, we get

$$I = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{1}{9}\left[5(0.4393) + 8(0.2474) + 5(0.1379)\right] = 0.5406.$$

The exact solution is I=0.5406.

8.8 Gauss-Chebyshev Integration Methods

Consider the formula

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)dx = \sum_{k=0}^{n} \alpha_k f(x_k).$$

In this case, all the nodes x_k and weight α_k are unknown. Consider the following cases.

One-point formula: let $n = 0$. The formula is given by

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)dx = \alpha_0 f(x_0).$$

The method has two unknowns α_0, x_0. Making the exact for $f(x) = 1, x$, we get

$$f(x)=1: \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = \alpha_0 \text{ or } \left[\sin^{-1}(x)\right]_{-1}^{1} = \alpha_0 \text{ or } \alpha_0 = \pi,$$

$$f(x)=x: \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}dx = \alpha_0 x_0 = 0 \text{ or } x_0 = 0, \alpha_0 = 0.$$

Hence, the method is given by

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)dx = \pi f(0).$$

The error constants is given by

$$C = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} x^2 dx = 2\int_{0}^{1} \frac{1}{\sqrt{1-x^2}} x^2 dx = 2\int_{0}^{\pi/2} \sin^2(\theta)d\theta = \frac{\pi}{3}.$$

Hence $R_1 = \frac{C}{2!} f''(\lambda) = \frac{\pi}{4} f''(\lambda), \quad -1 < \lambda < 1.$

Two-point formula: Let $n = 1$. The formula is given by

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)dx = \alpha_0 f(x_0) + \alpha_1 f(x_1).$$

The method has two unknowns α_0, x_0, α_1 and x_1. Making the exact for $f(x) = 1, x, x^2, x^3$, we get

$$f(x) = 1: \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \alpha_0 + \alpha_1 \quad \text{or} \quad \alpha_0 + \alpha_1 = \pi,$$

$$f(x) = x: 0 = \alpha_0 x_0 + \alpha_1 x_1,$$

$$f(x) = x^2: \frac{\pi}{2} = \alpha_0 x_0^2 + \alpha_1 x_1^2,$$

$$f(x) = x^3: 0 = \alpha_0 x_0^3 + \alpha_1 x_1^3.$$

Eliminating for α_0, we get $\alpha_0 = \alpha_1 = \pi / 2$.

And also $x_0 = \pm \frac{1}{\sqrt{2}}$ and $x_1 = \pm \frac{1}{\sqrt{2}}$.

There for, the two Point Gauss-Chybyshev methods is given by

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)dx = \frac{\pi}{2}\left[f(-\frac{1}{\sqrt{2}}) + f(\frac{1}{\sqrt{2}}) \right].$$

The error constants is given by

$$C = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} x^4 dx - \frac{\pi}{4}[\frac{1}{4} + \frac{1}{4}] = \int_{-\pi/2}^{\pi/2} \sin^4(\theta)dx - \frac{\pi}{4} = \frac{\pi}{8}.$$

Hence $R_4 = \frac{C}{4!} f^{(4)}(\lambda) = \frac{\pi}{192} f^{(4)}(\lambda), \quad -1 < \lambda < 1.$

Three-point formula: Let $n = 2$. The formula is given by

$$\int_{-1}^{1} f(x)dx = \alpha_0 f(x_0) + \alpha_1 f(x_1) + \alpha_2 f(x_2).$$

The method has two unknowns $\alpha_0, x_0, \alpha_1, x_1, \alpha_2$ and x_2. Making the exact for $f(x) = 1, x, x^2, x^3, x^4, x^5$, we get

$$f(x)=1: \quad \pi=\alpha_0+\alpha_1+\alpha_2,$$
$$f(x)=x: \quad 0=\alpha_0x_0+\alpha_1x_1+\alpha_2x_2,$$
$$f(x)=x^2: \quad \frac{\pi}{2}=\alpha_0x_0^2+\alpha_1x_1^2+\alpha_2x_2^2,$$
$$f(x)=x^3: \quad 0=\alpha_0x_0^3+\alpha_1x_1^3+\alpha_2x_3^3,$$
$$f(x)=x^4: \quad \frac{3\pi}{8}=\alpha_0x_0^4+\alpha_1x_1^4+\alpha_2x_3^4,$$
$$f(x)=x^5: \quad 0=\alpha_0x_0^5+\alpha_1x_1^5+\alpha_2x_3^5.$$

Eliminating for α_0, we get $\alpha_0=\frac{\pi}{3}=\alpha_2$ and $\alpha_1=\frac{\pi}{3}$.

And also $x_0=\pm\frac{\sqrt{3}}{2}$, $x_1=0$ and $x_2=\mp\frac{\sqrt{3}}{2}$.

There for, the three point Gauss- Chybyshev method is given by

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}f(x)dx=\frac{\pi}{3}\left[f(-\frac{\sqrt{3}}{2})+f(0)+f(\frac{\sqrt{3}}{2})\right].$$

The error constants is given by

$$C=\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}x^6dx-\frac{\pi}{3}[\frac{27}{64}+0+\frac{27}{64}].$$

Setting $x=\sin(\theta)$,

$$C=\int_{-\pi/2}^{\pi/2}\sin^6(\theta)d\theta-\frac{9\pi}{32}=2(\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2})=\frac{\pi}{32}.$$

Hence $R_6=\frac{C}{6!}f^{(6)}(\lambda)=\frac{\pi}{23040}f^{(6)}(\lambda),\ -1<\lambda<1.$

Example 8.22: Evaluate the integral $\int_{-1}^{1}(1-x^2)^{3/2}\cos(x)dx$, using Gauss-Chybyshev 1-point, 2-point and 3-point quadrature rules.

Solution: Using 1-point Gauss- Chybyshev rule, we get $I=\pi\ f(0)=3.14159.$

Using 2-point Gauss- Chybyshev rule, we get

$$I = \frac{\pi}{2}\left[f(-\frac{1}{\sqrt{2}}) + f(\frac{1}{\sqrt{2}}) \right] = \frac{\pi}{2}\left[2\frac{1}{4}\cos(\frac{1}{\sqrt{2}}) \right] = 0.59709.$$

Using 3-point rule, we

$$I = \frac{\pi}{3}\left[f(-\frac{\sqrt{3}}{2}) + f(0) + f(\frac{\sqrt{3}}{2}) \right] = \frac{\pi}{3}\left[2(\frac{1}{16})\cos(\frac{\sqrt{3}}{2}) + 1 \right] = 1.13200.$$

EXERCISES

1 a. Evaluate $\int_0^1 e^{-x^2} dx$, dividing the range into 4 equal part, Using:

i. Trapezoidal Rule , ii. Simpson's Rule (1/3).

b. Evaluate $\int_0^1 (1 + \frac{\sin(x)}{x}) dx$, use Simpson's (3/8) rule for *n*=6.

c. Use the Simpson's rule (1/3) to approximate $\int_1^5 \frac{x}{\sqrt{x+1}} dx$ with *n*=8.

2 a. Determine the step size *h* required in order for the Simpson's Rule (1/3) to approximate the integral $\int_0^8 x\sin(x)\,dx$, with an error of at most 10^{-4} .

b. Find the error bound for $\int_{-0.5}^{0.5} x\ln(x+2)dx$, approximate by the Simpson's rule (1/3).

c. Evaluate $\int_0^{0.5} \frac{x}{\cos(x)} dx$ with *n*=10, use Simpson's rule (1/3).

3. Evaluate the integral $\int_0^2 x^2 e^{-x^2} dx$, *h*=0.25 using i. Trapezoidal rule,

ii. Simpson (1/3) Rule.

4. Evaluate $\int_2^3 \frac{\cos(2x)}{1+\sin(x)} dx$ using the Gauss-Legender three points.

5. Evaluate $\int_0^2 \frac{(x^2+2x+1)}{1+(x+1)^4}dx$ using the Gauss-Legender three points.

6. Evaluate $\int_0^2 e^x\,dx$ using Romberg for h=1, compare with the exact solution.

7. Evaluate $\int_0^1 \frac{1}{1+e^{x^2}}dx$ using Romberg for *h*=1.

8. Evaluate $\int_0^1 \cos(2x)\,(1-x^2)^{\frac{-1}{2}}dx$ use Gauss-Chybeshev quadrature formula for three points.

9. Evaluate $\int_{-\infty}^{\infty} \frac{e^{-x^2}}{1+x^2}dx$ use Gauss-Hermite formula, n=2.