SUMMARY

This dissertation is related to the generalization of one of the most well-known unresolved problems in number theory, which is called Artin's primitive root conjecture. This celebrated conjecture asks for how many primes there are such that a prescribed rational number a has maximal multiplicative order modulo p, that is order $p-1$ [4]. Under the Generalized Riemann Hypothesis (GRH) it was proved by C. Hooley in 1967 [19]. The most significant results presented in this dissertation found in Chapters 3 and 4.

Let Γ be a finitely generated multiplicative subgroup of the rationals and $\mathrm{Supp}\,\Gamma$, the set of primes p divides no numerator nor denominator of any of the elements of Γ. For any prime $p \notin \mathrm{Supp}(\Gamma)$, we can consider the reduction group $\Gamma_p = \{\gamma \bmod p : \gamma \in \Gamma\} \subset \mathbb{F}_p^*$ and view it as a subgroup of the multiplicative group $\mathbb{F}_p^*$ of the finite field $\mathbb{F}_p$. Given a prescribed integer m, we are interested in the set $\pi_\Gamma(x, m)$ of primes p such that p is not in $\mathrm{Supp}(\Gamma)$ and the index of $\mathbb{F}_p$ in $\mathbb{F}_p^*$ equals m. Under the Generalized Riemann Hypothesis it is known that this set of primes has a natural density $\rho(\Gamma, m)$ that is given by

$$\rho(\Gamma, m) = \sum_{k \geq 1} \frac{\mu(k)}{[\mathbb{Q}(\zeta_{mk}, \Gamma^{1/mk}) : \mathbb{Q}]},$$

with $\zeta_d = e^{2\pi i/d}$ and $\Gamma^{1/d}$ the set of real numbers α such that α^d is in Γ.

The results of this dissertation can be explained in two parts. The first part of our result we explicitly evaluates $\rho(\Gamma, m)$ as a rational number times the Euler product

$$\prod_{\ell} \left(1 - \frac{1}{(\ell - 1)|\Gamma.\mathbb{Q}^{*\ell}/\mathbb{Q}^{*\ell}|}\right)$$

where the product ranges over the odd primes ℓ not dividing m. In earlier work of Pappalardi and Susa [46] where they had established this result in case Γ contains only positive rational numbers, but we extend their work to Γ, which

may contain negative rational numbers as well. We work out the special case, where $\Gamma = \langle -1, a \rangle$. We are also interested in determining under which conditions $\rho(\Gamma, m) = 0$ and under which conditions $\pi_\Gamma(x, m)$ is a finite set. The proofs require algebraic number theory in order to evaluate the field degrees of the Kummer extensions involved, and in addition fairly technical manipulations with in finite sums having an argument that is nearly a multiplicative function. Moreover, the results in this part are backed up by a lot of numerical data that agrees very well with the results from the theory.

The second part of our results, we show that, given any natural number m, the primes p such that m dividing Γ_p have a natural density and are evaluated by bringing it into Euler product form. We start out by writing down an inclusion-exclusion sum for the density in which each individual density follows from the prime ideal theorem and involves a degree of a number field. This is a standard calculation. The next step is then to evaluate the resulting expression and bring it into Euler product form. Here, as so often happens, the prime 2 (the oddest of primes) plays a special-complicating- role. After an involved, but rather standard, calculation, we then end up with a relatively clean theorem. We then go on to show that in the special case $\Gamma = \langle a \rangle$, with a a rational number the density obtained coincides with a formula computed by Moree (who provided the most generic formula in this particular instance). Further, we present 4 tables with experimental verification of their results.